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FROM THE "EDUCATIONAL TIMES,"

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Solved Questions.

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CORRIGENDUM.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2352. (Prof. SYLVESTER, F.R.S.)—We may use P_*Q to denote the third point in which the right line PQ meets a given cubic; P_*Q_*R to denote the third point in which the line joining the one last named and R meets the cubic, and so on. Thus P_*P will denote the tangential or point in which the tangent at P meets the given cubic, and $[P_*P]_*[P_*P]$ will denote the second tangential, *i.e.*, the tangential to the tangential at P.

1. Prove that $[P_*P]_*[P_*P] = I_*P_*[P_*P]_*P_*I$, where I is any point of inflexion in the given curve.

2. Obtain a function of P, I which shall express the point in which the curve is cut by a conic having five-point contact with it at P.

Solution by Professor NASH, M.A.

(1) This theorem may be stated as follows:—If T_1, T_2 denote the first and second tangentials of a point P on a cubic, I a point of inflexion, and if IP meet the curve in Q, QT_1 meet the curve R, RP meet the curve in S, then SI will pass through T_2.

IP and RT_1 are coresidual, Q being residual to both pairs of points. But the tangents at I and P may be considered as a conic through the six points I, I, I, P, P, T_1; therefore the four points I, I, P, T_1 are residual to IP, and therefore also to RT_1; therefore I, I, T_1, T_1, P, R lie upon a conic, and every conic through the four points I, I, T_1, T_1 will meet the cubic again in two points, the line joining which will pass through the coresidual of the four points, *i.e.*, S. But the tangents at I and T_1 form such a conic, and the two points are I, T_2; therefore I, T_2, S are collinear.

(2) The required point is the intersection of PT_2 with the cubic (SALMON's *Higher Plane Curves*, Art. 155), and this may be expressed in Prof. Sylvester's notation as $P_*\{(P_*P)_*(P_*P)\}$ or $(P_*P)_*(P_*P)_*P$, and therefore by (1) the same point is represented by $I_*P_*(P_*P)_*P_*I_*P$.

9307. (Professor GENESE, M.A.)—In the ordinary conical projection of one given plane on another from a given vertex, prove that there is a

point in space, other than the vertex, at which every line and its projection subtend equal angles.

Solution by the PROPOSER.

Draw a plane p through the vertex V and the line of intersection l of the given planes; through l draw the plane k which is harmonically conjugate to p with respect to the other planes; from V draw VO perpendicular to k; then O is the point in question. Let any straight line through V meet the given planes in P, P′ and k in L, then (VPKLP′) is harmonic, and RVOL is a right angle, therefore OP, OP′ are equally inclined to OL, *and they are in a plane normal to k*. Similarly for a second line VQQ; whence, by *symmetry*, angle POQ = angle P′OQ′.

9272. (Professor IGNACIO BEYENS.)—Résoudre en nombres entiers et positifs l'équation $x^2 - yz \pm a^2 = 0$.

Solution by R. W. D. CHRISTIE, M.A.; E. RUTTER; *and others.*

We have $x^2 - yz = \pm a^2 = (x - yz/n)^2$ say, whence we get

$$n^2 - 2nx + yz = 0; \quad \text{therefore } x^2 - yz = \pm a^2 = \pm(n-x)^2;$$

therefore $x = \pm(n-a)$; and hence, easily, $y = n$ and $z = n - 2a$, where n and a may be any integers, regard being had to the signs.

9320. (ISABEL MADDISON.)—Four lines, p, q, r, s, in a plane are cut by a line a. Prove that the point $a\left[(pq)\left\{(as\,.\,rq)(ar\,.\,sp)\right\}\right]$ is unchanged when any of the letters p, q, r, s are interchanged. [In the above complex symbol the combination of two line symbols represents a point, and the combination of two point symbols represents a line.]

Solution by Prince de POLIGNAC; F. R. J. HERVEY; *and others.*

The equations of the lines being $a = 0$, &c., assume $p = a + lr + ms$, $q = a + l'r + m's$. The equations to $as\,.\,rq$ and $ar\,.\,sp$ are respectively $a + m's = 0$ and $a + lr = 0$. To find the line joining their intersection to pq, assume the forms $a + lr - \lambda(a + m's) = 0$, $p - \mu q = 0$, and equate ratios of coefficients; we find $\mu = lm/(l'm')$, and the line is

$$(l'm' - lm)\,a + ll'(m' - m)\,r + mm'(l' - l)\,s = 0 \quad \text{..............(1)}.$$

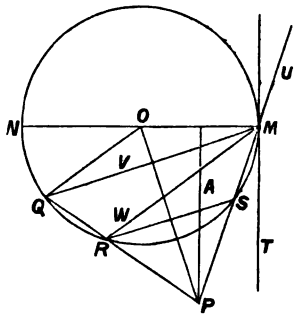
To find the result of interchanging r, s or p, q, we either interchange l, m or displace the accents. These changes leave the line (1) unaltered; hence the points pq, $(as\,.\,rq)\,(ar\,.\,sp)$, $(ar\,.\,sq)\,(as\,.\,rp)$ are collinear. Thus the permutations arrange themselves in six groups, to each of which corresponds a single line passing through one of the intersections of p, q, r, s.

Interchange s, q; the equations to $aq\,.\,rs$ and $ar\,.\,qp$ are evidently $l'r + m's = 0$ and $(m' - m)\,a + (lm' - l'm)\,r = 0$; the corresponding line through sp is $\quad l'(m' - m)\,a + ll'(m' - m)\,r + mm'(l' - l)\,s = 0$ ....(2).

It follows that the lines through any two points, such as pq and ps, having a common line p, intersect on a; which proves the theorem. The equations of the lines through pr, qr, and qs are derived from (2) by interchanges. [The point on a is, by BRIANCHON's theorem, the point of tangency with the conic that touches the five lines a, p, q, r, s, &c.]

9361. (F. R. J. HERVEY.)—A line A bisects at right angles the radius OM of a circle (centre O); three lines U, V, W, passing through M, rotate uniformly with angular velocities as $1 : -1 : -2$, and cut respectively A in P, and the circle in Q, R; V and W passing through O at the instant that U becomes a tangent. Prove that P, Q, R are always collinear, and PQ . QR constant.

Solution by R. F. DAVIS, M.A.; D. BIDDLE; *and others.*



Let U meet the circle in S and QR in P. Since the angles QMN, QMR, SMT are (by hypothesis) equal, so also are the arcs QN, QR, SM. Hence OP bisects at right angles the parallels QM, RS; and therefore the angles POM, PMO are equal, being the complements of equal angles, and P lies on A.

Since $\angle QOR = 2QMR$

$= OPM = OPR,$

QO touches the circumcircle of OPR and

$$QO^2 = PQ \,.\, QR.$$

7178. (W. J. C. SHARP, M.A.)—If three concyclic foci of a bicircular quartic, or circular cubic, be given, and also a tangent and its point of contact, determine the curve.

Solution by Professors MATZ, M.A.; NASH, M.A.; *and others.*

Let A, B, C be the three given points, P the point of contact of the given tangent. The quartic (or cubic) is the envelope of a circle whose centre moves on a certain conic through A, B, C, and which cuts ortho-

gonally the circle ABC; the curve will therefore be completely determined if the conic can be determined.

Take P′ the inverse of P with respect to the circle ABC, then P′ is also on the curve, and the line which bisects PP′ at right angles touches the conic at the centre of the variable circle which touches the curve at P and P′. This tangent is therefore known, and its point of contact is its intersection with a perpendicular to the given tangent at P. Hence four points are given on the conic, and the tangent at one of them, so that the conic is completely determined.

9271. (Professor DE WACHTER.)—A straight rod is divided at random into four parts; prove that it is an even chance that these parts may be the sides of any quadrilateral.

Solution by ARTEMAS MARTIN, LL.D.

Denote the parts by x, y, z, and $a-x-y-z$, a being the length of the rod. The following conditions must be satisfied, viz., $x<\frac{1}{2}a$, $y<\frac{1}{2}a$, $z<\frac{1}{2}a$, $x+y+z>a-x-y-z$.

The required probability will be

$$p = \iiint dx\,dy\,dz \Big/ \iiint dx\,dy\,dz.$$

In N, x may have any value from 0 to $\frac{1}{2}a$; y may have any value from 0 to $\frac{1}{2}a$; z may have any value from $\frac{1}{2}a-x-y$ to $\frac{1}{2}a$ when y is less than $\frac{1}{2}a-x$, and any value from 0 to $a-x-y$ when y is greater than $\frac{1}{2}a-x$. In D, x may have any value from 0 to a; y may have any value from 0 to $a-x$; z may have any value from 0 to $a-x-y$. Hence

$$p = \int_0^{\frac{1}{2}a}\left[\int_0^{\frac{1}{2}a-x}\int_{\frac{1}{2}a-x-y}^{\frac{1}{2}a} dy\,dz + \int_{\frac{1}{2}a-x}^{\frac{1}{2}a}\int_0^{a-x-y} dy\,dz\right]dx \Big/ \int_0^a\int_0^{a-x}\int_0^{a-x-y} dx\,dy\,dz$$

$$= \frac{6}{a^3}\int_0^{\frac{1}{2}a}\left[\int_0^{\frac{1}{2}a-x}\int_{\frac{1}{2}a-x-y}^{\frac{1}{2}a} dy\,dz + \int_{\frac{1}{2}a-x}^{\frac{1}{2}a}\int_0^{a-x-y} dy\,dz\right]dx$$

$$= \frac{6}{a^3}\int_0^{\frac{1}{2}a}\left[\int_0^{\frac{1}{2}a-x}(x+y)\,dy + \int_{\frac{1}{2}a-x}^{\frac{1}{2}a}(a-x-y)\,dy\right]dx$$

$$= \frac{6}{a^3}\int_0^{\frac{1}{2}a}\left(\tfrac{1}{8}a^2 + \tfrac{1}{2}ax - x^2\right)dx = \frac{1}{2}.$$

[If we take a regular tetrahedron, the altitude of which is the length (l) of the rod, and from any interior point draw perpendiculars to the faces, then the sum of these four perpendiculars will be $= l$. Any interior point represents (by those perpendiculars) a distinct chance of division of the rod; and the favourable points are situated so as to have each of their distances $< \frac{1}{2}l$. If, therefore, planes be drawn parallel to the faces, and equi-distant from each vertex and the opposite face, it is easy to see that the favourable points are included between those four planes and the faces of the tetrahedron, and they form a regular octahedron. Hence the probability will be the ratio of the Octahedron to the Tetrahedron, that is to say, $\frac{1}{2}$.]

8132. (W. J. Johnston, M.A.)—Prove that, if the section of a quadric by a plane is given, and also a straight line in that plane; then, if through this line a plane can be drawn to cut the quadric in a circular section whose radius is also given, the locus of the centre of this circular section is a circle in a plane perpendicular to the given plane.

Solution by G. G. Storr, M.A.; A. Gordon; *and others.*

Let the given line be chosen as axis of y, the given plane as plane of xy, and a normal to it as axis of z. If

$$ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy+2a''x+2b''y+2c''z+d=0$$

is the quadric, then $\frac{a}{d},\ \frac{b}{d},\ \frac{c'}{d},\ \frac{a''}{d},\ \frac{b''}{d}$ are known, since the section $z=0$ is known.

Let the plane $z=0$ revolve about y, through an angle θ till it forms the section required, so that $x=\xi\cos\theta-\zeta\sin\theta$, $z=\xi\sin\theta+\zeta\cos\theta$. Substituting after putting $\zeta=0$, we have

$$\xi^2(a\cos^2\theta+c\sin^2\theta+2b'\sin\theta\cos\theta)+by^2+2\xi y(a'\sin\theta+c'\cos\theta)+d$$
$$+2\xi(a''\cos\theta+c''\sin\theta)+2yb''=0.$$

In order that this may be a circle, it is necessary and sufficient that

$$a\cos^2\theta+c\sin^2\theta+2b'\sin\theta\cos\theta=b,\quad a'\sin\theta+c'\cos\theta=0;$$

the coordinates of the centre are given by

$$y_1=-\frac{b''}{b}\ \text{(a constant)},\quad \xi_1=-\frac{a''\cos+c''\sin}{b},$$

and $$(\text{radius})^2=-d/b+\xi_1^2+y_1^2,$$

therefore ξ_1 is a constant. But $\xi_1=x_1\cos\theta+z_1\sin\theta=\lambda$ suppose, therefore $x_1^2+z_1^2=\lambda^2$, $y_1=$ constant, is the locus required.

[The conditions amongst the variable coefficients c, a', b', c'', in order that the circular section can be obtained of given radius, are that the equation

$$(a-b)\cos^2\theta+(c-b)\sin^2\theta+2b'\sin\theta\cos\theta=0,$$

must give real roots for $\tan\theta$, or $b'^2\geq(a-b)(c-b)$ ....(1),

and $\tan\theta=-c'/a'$ must satisfy the above, or

$$(a-b)\,a'^2+(c-b)\,a'^2-2b'ac'=0\quad\text{.....................}\ (2),$$

also the condition that $(\text{radius})^2+d/b-(b''/b)^2\geq0$ ....(3).]

9324. (Rev. T. C. Simmons, M.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\frac{dx}{(a^2+b^2\tan^2x)^n}=\frac{\pi}{4a^3}\cdot\frac{2a^3-3a^2b+b^3}{(a^2-b^2)^2},\quad \frac{\pi}{16a^5}\cdot\frac{8a^5-15a^4b+10a^2b^3-3b^5}{(a^2-b^2)^3},$$

when $n=2, 3$; and deduce, if possible, a general formula for this type of definite integral.

Solution by Prof. WOLSTENHOLME, Sc.D.; J. W. SHARPE, M.A.; *and others.*

Writing $a^2=p$, $b^2=q$, let $U_n=\int_0^{\frac{1}{2}\pi}\frac{dx}{(p+q\tan^2x)^n}$, then

$$U_n=-\frac{1}{n-1}\frac{d}{dp}(U_{n-1})=\frac{1}{n-1\,.\,n-2}\frac{d^2}{dp^2}(U_{n-2}),\text{ and so on,}$$

or,
$$U_n=\frac{(-1)^{n-1}}{n!}\frac{d^{n-1}}{dp^{n-1}}(U_1);$$

and
$$U_1=\int_0^{\frac{1}{2}\pi}\frac{dx}{p+q\tan^2x}=\int_0^{\infty}\frac{dz}{(1+z^2)(p+qz^2)}=\frac{1}{p-q}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{1+z^2}-\frac{q}{p+qz^2}\right)dz$$
$$=\tfrac{1}{2}\pi\frac{1}{p-q}\left(1-\sqrt{\frac{q}{p}}\right)=\tfrac{1}{2}\pi\left(\frac{1}{p+\sqrt{pq}}\right)=\frac{\pi}{2\sqrt{q}}\left(\frac{1}{\sqrt{p}}-\frac{1}{\sqrt{p}+\sqrt{q}}\right).$$

Hence
$$U_n=\tfrac{1}{2}\pi\frac{1.3.5\ldots2(n-3)}{2.4.6\ldots2(n-2)}\frac{1}{p^{n-\frac{1}{2}}q^{\frac{1}{2}}}+\frac{\pi}{2\sqrt{q}}\frac{(-1)^n}{n!}\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{\sqrt{p}+\sqrt{q}}\right)$$
$$=\frac{\pi}{2a^{2n-1}b}\frac{1.3.5\ldots2(n-3)}{2.4.6\ldots2(n-2)}$$
$$+\frac{\pi}{2b}\frac{(-1)^n}{2.4.6\ldots2(n-2)}\left(\frac{1}{a}\frac{d}{da}\right)^{n-1}\left(\frac{1}{a+b}\right);$$

where a, b are positive. Thus

$$U_2=\frac{\pi}{4}\frac{1}{a^3b}-\frac{\pi}{4b}\frac{1}{(a+b)^2}=\frac{\pi}{4}\frac{2a+b}{a^3(a+b)^2}=\frac{\pi}{4}\frac{2a^3-3a^2b+b^3}{a^3(a^2-b^2)^2};$$

$$U_3=\frac{3\pi}{16}\frac{1}{a^5b}-\frac{\pi}{16b}\frac{1}{a}\left(\frac{1}{a^2(a+b)^2}+\frac{2}{a(a+b)^3}\right)=\frac{\pi}{16a^5b}\left(3-\frac{a^2(3a+b)}{(a+b)^3}\right)$$
$$=\frac{\pi}{16}\frac{8a^2+9ab+3b^2}{a^5(a+b)^3}=\frac{\pi}{16}\frac{8a^5-15a^4b+10a^2b^3-3b^5}{a^5(a^2-b^2)^3}.$$

In the same way the value of the definite integral

$$\int_0^{\frac{1}{2}\pi}\frac{\tan^{2m}x}{(p+q\tan^2x)^n}dx,$$

where m, n, and $n-m$ are positive integers (or zero), will be found to be

$$\frac{\pi}{2}\frac{(-1)^{n-1}}{n!}\left(\frac{d}{dp}\right)^{n-m-1}\left(\frac{d}{dq}\right)^m\left(\frac{1}{p+\sqrt{pq}}\right).$$

(Obviously, if p, q be of opposite sign, the integral will be infinite.)

Another method of evaluating U_n is:—Put $\sqrt{q}\tan x=\sqrt{p}\tan z$, then

$$p+q\tan^2x=p\sec^2z,\quad \sqrt{q}\sec^2x\,dx=\sqrt{p}\sec^2z\,dz,$$

and the limits 0, $\frac{1}{2}\pi$ are unchanged, so that

$$U_n=\frac{1}{p^n}\sqrt{\frac{p}{q}}\int_0^{\frac{1}{2}\pi}\frac{dz}{(1+p/q\tan^2z)(1+\tan^2z)^{n-1}}=\frac{\sqrt{pq}}{p^n}\int_0^{\frac{1}{2}\pi}\frac{\cos^{2n}z\,dz}{p\sin^2z+q\cos^2z};$$

or
$$U_n=\frac{\sqrt{pq}}{p^n(p-q)^n}\int_0^{\frac{1}{2}\pi}\left(\frac{p^n}{p\sin^2z+y\cos^2z}-\frac{p^n-(p-q)^n\cos^{2n}z}{p-(p-q)\cos^2z}\right)dz$$
$$=\frac{\sqrt{pq}}{p^n(p-q)^n}\int_0^{\frac{1}{2}\pi}\left\{\frac{p^n\sec^2z\,dz}{q+p\tan^2z}-p^{n-1}-p^{n-2}(p-q)\cos^2z\right.$$
$$\left.-p^{n-3}(p-q)^2\cos^4z-\ldots\text{ to }n\text{ terms}\right\}dz$$

$$=\frac{\pi}{2}\,\frac{\sqrt{pq}}{p^n(p-q)^n}\left\{\frac{p^n}{\sqrt{pq}}-p^{n-1}-p^{n-2}(p-q)\frac{1}{2}-p^{n-3}(p-q)^2\frac{1\,.\,3}{2\,.\,3}\right.$$
$$\left.-\ldots \text{ to } n \text{ terms}\right\};$$

or
$$\int_0^{\frac{1}{2}\pi}\frac{dx}{(a^2+b^2\tan^2 x)^n}=\frac{\pi}{2a^{2n-1}}\times$$
$$\left\{\frac{a^{2n-1}-b\left[a^{2n-2}+\frac{1}{2}a^{2n-4}(a^2-b^2)+\frac{1\,.\,3}{2\,.\,4}a^{2n-6}(a^2-b^2)^2+\ldots \text{ to } n \text{ terms}\right]}{(a^2-b^2)^n}\right\}.$$

The result in this form may always be reduced by the factors $(a-b)^n$, which must be a factor of the numerator, otherwise the integral would be infinite when $a=b$. Putting $n=2$, $n=3$, we get the results.

Equating the two values for $\int_0^{\frac{1}{2}\pi}\frac{dx}{(p+q\tan^2 x)^n}$, we get an expression for $\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{p+\sqrt{pq}}\right)$, and since this $=\frac{1}{\sqrt{q}}\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{\sqrt{p}}-\frac{1}{\sqrt{p}+\sqrt{q}}\right)$ we also get a finite series for $\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{\sqrt{p}+\sqrt{q}}\right)$.

The integral $\int_0^{\frac{1}{2}\pi}\frac{\tan^{2m} x}{(a^2+b^2\tan^2 x)^n}dx$ $(n>m)$ may be evaluated in the same way, being equal to

$$\frac{\sqrt{pq}}{q^m p^{n-m}}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2m} z\cos^{2n-2m} z\,dz}{p\sin^2 z+q\cos^2 z}dz$$
$$=\frac{\sqrt{pq}}{p^{n-m}q^m}\,\frac{1}{(p-q)^{n-m}}\int_0^{\frac{1}{2}\pi}\left(\frac{p^{n-m}\sin^{2m} z}{p\sin^2 z+q\cos^2 z}\right)-p^{n-m-1}-p^{n-m-1}(p-q)\cos^2 z$$
$$=\frac{\sqrt{pq}}{p^{n-m}q^m(p-q)^{n-m}}\int_0^{\frac{1}{2}\pi}\sin^{2m} z\left\{\frac{p^{n-m}}{p\sin^2 z+q\cos^2 z}-p^{n-m-1}-p^{n-m-2}\cos^2 z\right.$$
$$\left.-p^{n-m-3}\cos^4 z-\ldots \text{ to } (n-m) \text{ terms}\right\}dz,$$

in which the value of each term may be written down at once, with the exception of the first; and

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^{2m} z\,dz}{p\sin^2 z+q\cos^2 z}=\frac{1}{(p-q)^m}\left\{\int_0^{\frac{1}{2}\pi}\frac{(p-q)^m\sin^{2m} z\pm q^m}{(p-q)\sin^2 z+q}dz\mp\frac{q^m}{\sqrt{pq}}\,\frac{\pi}{2}\right\},$$

which gives the result as a series of m terms.

Thus
$$\int_0^{\frac{1}{2}\pi}\frac{\tan^2 x}{(p+q\tan^2 x)^2}dx=\frac{\pi}{4}\,\frac{1}{\sqrt{pq}(\sqrt{p}+\sqrt{q})^2},$$
$$\int_0^{\frac{1}{2}\pi}\frac{\tan^4 x}{(p+q\tan^2 x)^3}dx=\frac{\pi}{16}\left\{\frac{1}{\sqrt{pq^3}(\sqrt{p}+\sqrt{q})^2}+\frac{2}{\sqrt{pq^2}(\sqrt{p}+\sqrt{q})^3}\right\}$$
$$=\frac{\pi}{16}\,\frac{\sqrt{p}+3\sqrt{q}}{q\sqrt{pq}(\sqrt{p}+\sqrt{q})^3};$$

but I think the former method preferable.

9316. (Professor WOLSTENHOLME, Sc.D.)—In any curve OM $= x$, MP $= y$ are coordinates of a point P, MQ is drawn perpendicular to the tangent at P and bisected by it; prove that the arc σ of the locus of Q is given by the equation

$$\frac{d\sigma}{d\theta} = \pm\left(2y - \frac{dx}{d\theta}\right), \text{ where } \frac{dy}{dx} = \tan\theta; \text{ and that}$$

(1) when $x^2+y^2 = a^2$, the whole arc of the locus of Q $= 12a$;
(2) when $y^2 = 4ax$, the arc from the vertex $= x + 2a\log(1 + x/a)$;
(3) when $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a > b)$, the whole arc $= 4a\left(1 + \dfrac{1-e^2}{e}\log\dfrac{1+e}{1-e}\right)$;
(4) .... $= (a < b)$, ... $= 4b\left\{(1-e^2)^{\frac{1}{2}} + 2/e\sin^{-1}e\right\}$;
(5) when $x = a(2\phi + \sin 2\phi)$, $y = a(1+\cos 2\phi)$, $\sigma = 2x$;
(6) when $x = a(2\phi + \sin 2\phi)$, $y = a(1-\cos 2\phi)$, the locus of Q is a cycloid of half the linear dimensions and having the same tangent at the vertices;
(7) when the curve is such that the radius of curvature is n times the normal at P terminated by the axis of x, the arc $= \pm(n-2)/n \,.\, x$, n being any constant number.

Solution by J. W. SHARPE, M.A.

Let ξ, η be the coordinates of Q; then

$$\xi = x - y\sin 2\theta, \quad \eta = y(1+\cos 2\theta), \text{ and } dy = dx \,.\, \tan\theta,$$

therefore $\quad d\xi = (dx - 2y\,d\theta)\cos 2\theta, \quad d\eta = (dx - 2y\,d\theta)\sin 2\theta,$

therefore $\quad d\sigma = \pm(dx - 2y\,d\theta).$

(1) $x^2+y^2=a^2$; therefore $\tan\theta = \pm\dfrac{x}{y}$, and $d\theta = -\dfrac{dx}{(a^2-x^2)^{\frac{1}{2}}}$,

therefore $\quad \sigma = 4\displaystyle\int_0^a 3dx$; therefore $\sigma = 12a$.

(2) $y^2=4ax$; therefore $\tan\theta = \dfrac{2a}{y}$; and $d\theta = -\dfrac{\sqrt{a}\,.\,dx}{2(x+a)\sqrt{x}}$;

therefore $\quad \sigma = \displaystyle\int_0^x \left(\frac{2a}{x+a} + 1\right)dx = x + 2a\log\left(1 + \frac{x}{a}\right).$

(3) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Put $x = a\cos\phi$, $y = b\sin\phi$; then

$$d\theta = \frac{ab\,d\phi}{a^2\sin^2\phi + b^2\cos^2\phi}; \text{ therefore } \frac{d\sigma}{a} = \left\{\frac{2b^2\sin\phi}{a^2\sin^2\phi + b^2\cos^2\phi} + \sin\phi\right\}d\phi;$$

therefore
$$\frac{\sigma}{4a} = 2b^2\int_0^{\frac{1}{2}\pi}\left\{\frac{d\cos\phi}{a^2-(a^2-b^2)\cos^2\phi} + \sin\phi\,d\phi\right\}$$
$$= 1 + \frac{1-e^2}{e}\log\frac{1+e}{1-e}.$$

(4)
$$\frac{\sigma}{4a} = 2b^2\int_0^{\frac{1}{2}\pi}\left\{\frac{d\cos\phi}{a^2+(b^2-a^2)\cos^2\phi} + \sin\phi\,d\phi\right\}$$
$$= \frac{2}{e(1-e^2)^{\frac{1}{2}}}\tan^{-1}\frac{e}{(1-e^2)^{\frac{1}{2}}} + 1;$$

therefore
$$\frac{\sigma}{4b} = \frac{2}{e}\sin^{-1}e + (1-e^2)^{\frac{1}{2}}.$$

(5) $x = a(2\phi + \sin 2\phi)$, $y = a(1+\cos 2\phi)$; then $\tan\theta = -\tan\phi$; therefore $\theta + \phi = \pi$; therefore $\dfrac{\sigma}{a} = \int_0^{\phi} 4(1+\cos 2\phi)\,d\phi = 2x$.

(6) $x = a(2\phi + \sin 2\phi)$, $y = a(1-\cos 2\phi)$; then $\tan\theta = \tan\phi$; therefore $\theta = \phi$; therefore $\xi = \frac{1}{2}a(4\phi + \sin 4\phi)$, $\eta = \frac{1}{2}a(1-\cos 4\phi)$.

(7) $\dfrac{(1+p^2)^{\frac{3}{2}}}{dp/dx} = ny\sec\theta$, where $p = \tan\theta$; therefore $\dfrac{dx}{d\theta} = ny$; therefore $\sigma = \int_0^{\theta}(n-2)\,y\,d\theta = \int_0^{x}\dfrac{n-2}{n}\,dx$; therefore $\sigma = \pm\dfrac{n-2}{n}\,x$.

8954. (W. J. C. Sharp, M.A.)—If seven tangents to a cuspidal cubic (or tricuspidal quartic) be given, and a conic be described to touch any four of those, the conic which touches the other three given tangents and the two remaining common tangents of the first conic and the curve, will always touch a fixed tangent to the curve.

Solution by Professors Nash, M.A.; Sarkar, M.A.; *and others.*

A cuspidal cubic (or bicuspidal quartic) being of the third class, its reciprocal is a cubic, and the reciprocal theorem may be stated as follows:—Given seven points A, B, C, D, E, F, G on a cubic, if through four of them, A, B, C, D, a conic be described meeting the cubic again in P, Q, the conic which passes through P, Q, and the other three points E, F, G, will pass through another fixed point H in the cubic. This follows at once from the well-known theorem that PQ passes through a fixed point R in the cubic, the coresidual of the four points A, B, C, D, and also of the four points E, F, G, H. Therefore, &c.

9128. (M. F. J. Mann, M.A.)—Find the sum of all numbers less than n and prime to it is divisible by n.

Solution by the Proposer.

If a is a prime to n, $n-a$ is also prime to n; hence, therefore, all the numbers less than n and prime to it, may be arranged in pairs, the sum of each pair being n.

9227. (W. J. C. Sharp, M.A.)—Show that (1) $1.2.3...n^p$ is divisible by (n) to the power of $(n^p-1)/(n-1)$; and (2) when (n) is a prime this is the highest power of (n) which will measure it.

Solution by Professor IGNACIO BEYENS.

Si nous faisons $n^p = \mathrm{N}$, la plus grande puissance d'un nombre premier contenue dans le produit $1 \,.\, 2 \,.\, 3 \ldots \mathrm{N}$ est

$$\frac{\mathrm{N}}{a} + \frac{\mathrm{N}'}{a} + \frac{\mathrm{N}''}{a} + \ldots.$$

N' étant $= \dfrac{\mathrm{N}}{a}$, et ainsi de suite ; mais si (a) n'est pas nombre premier alors

$$\frac{\mathrm{N}}{a} + \frac{\mathrm{N}'}{a} + \frac{\mathrm{N}''}{a} + \ldots$$

sera l'exposant d'une puissance de (a) qui divisera $1 \,.\, 2 \,.\, 3 \ldots \mathrm{N}$. Cela posé, comme $\mathrm{N} = n^p$,

$$\frac{n^p}{n} + \frac{n^{p-1}}{n} + \ldots + \left(\frac{n}{n} = 1\right) = n^{p-1} + n^{p-2} + \ldots + 1 = \frac{n^p - 1}{n-1}$$

sera le degré d'une puissance de (n) qui sera facteur de $1 \,.\, 2 \,.\, 3 \ldots n^p$; et si (n) est premier, $(n^p - 1) \,/\, (n-1)$ sera le nombre plus grand de fois que $(1 \,.\, 2 \,.\, 3 \ldots n^p)$ contiendra au facteur (n).

8742. (R. KNOWLES, B.A. Suggested by Quest. 8521.)—The circle of curvature is drawn at a point P of a parabola, PQ is the common chord; if O, O′ be the poles of chords of the parabola, normal to the parabola at P and Q respectively, and if M, N, R, T be the mid-points of OO′, OQ, O′P, PQ respectively, prove (1) that the lines MT, NR intersect at their mid-points in the directrix, (2) that OP, O′Q are bisected by the directrix.

Solution by Rev. T. R. TERRY, M.A.; Professor NASH, M.A.; *and others.*

Let the coordinates of P be ap^2, $2ap$; then equation to PQ is $x - ap^2 + p\,(y - 2ap) = 0$; therefore coordinates of Q are $(9ap^2, -6ap)$; normal at P is $y - 2ap + p\,(x - ap^2) = 0$; therefore coordinates of O, O′, M, N, R, T are

$$(-2a - ap^2,\ -2ap^{-1}),\quad (-2a - 9ap^2,\ \tfrac{2}{3}ap^{-1}),\quad (-2a - 5ap^2,\ -\tfrac{2}{3}ap^{-1}),$$
$$(-a + 4ap^2,\ -ap^{-1} - 3ap),\quad (-a - 4ap^2,\ \tfrac{1}{3}ap^{-1} + ap),\quad (5ap^2,\ -2ap);$$

therefore coordinates of middle points of MT and NR are

$$(-a,\ -\tfrac{1}{3}ap^{-1} - ap),$$

whence (1). Also abscissa of middle points of OP and O′Q is $-a$, whence (2).

8463. (J. C. STEWART, M.A.)—Solve completely the equations
$x + 2y - xy^2 + \sqrt{3}\,(1 - 2xy - y^2) = y + 2x - x^2y + (2 + \sqrt{3})(1 - 2xy - x^2) = 0$;
and show that one system of values is $x = \pm\frac{1}{3}\sqrt{3}$, $y = 1$ and $\sqrt{3} - 2$.

Solution by Professor SARKAR, M.A.; BELLE EASTON; *and others.*

The first equation may be put into the form

$$-\sqrt{3} = \frac{x(1-y^2)+2y}{1-y^2-2xy} = \left(x+\frac{2y}{1-y^2}\right)\bigg/\left(1-x\,\frac{2y}{1-y^2}\right),$$

or
$$m\pi + \tfrac{2}{3}\pi = \tan^{-1}x + 2\tan^{-1}y.$$

Similarly, from the second equation,

$$n\pi + \tfrac{7}{12}\pi = 2\tan^{-1}x + \tan^{-1}y,$$

therefore $\tan^{-1}x = (2n-m)\,\frac{1}{3}\pi + \frac{1}{6}\pi$, and $\tan^{-1}y = (2m-n)\,\frac{1}{3}\pi + \frac{1}{4}\pi$.

8095. (H. G. DAWSON, B.A.)—If a, b, c be the axes of a quadric having the tetrahedron of reference for a self-conjugate tetrahedron, $(\xi, \eta, \zeta, \theta)$ the tetrahedral coordinates of the centre of the quadric, and $(\lambda_1, \mu_1, \nu_1, \pi_1)$, $(\lambda_2, \mu_2, \nu_2, \pi_2)$, $(\lambda_3, \mu_3, \nu_3, \pi_3)$ the tangential coordinates of its principal planes; prove that (1)

$$-a^2 = \lambda_1^2\xi + \mu_1^2\eta + \nu_1^2\zeta + \pi_1^2\theta, \quad -b^2 = \lambda_2^2\xi + \mu_2^2\eta + \nu_2^2\zeta + \pi_2^2\theta,$$
$$-c^2 = \lambda_3^2\xi + \mu_3^2\eta + \nu_3^2\zeta + \pi_3^2\theta;$$

and hence (2), if a tetrahedron be self-conjugate with respect to a sphere of radius R and centre O, show

$$-R^2(ABCD) = \lambda^2(OBCD) + \mu^2(OCDA) + \nu^2(ODAB) + \pi^2(OABC),$$

where A, B, C, D are the vertices of the tetrahedron, λ, μ, ν, π the perpendiculars from A, B, C, D on any plane through O, and (ABCD), &c. are the volumes of the tetrahedra.

Solution by the PROPOSER; A. GORDON; *and others.*

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the four vertices of the tetrahedron, and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ the quadric. Then we have

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=1, \quad \frac{x_1x_3}{a^2}+\frac{y_1y_3}{b^2}+\frac{z_1z_3}{c^2}=1, \quad \frac{x_1x_4}{a^2}+\frac{y_1y_4}{b^2}+\frac{z_1z_4}{c^2}=1.$$

Hence, if
$$(A_1B_2C_3D_4) \equiv \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$\left.\begin{aligned} \frac{x_1}{a^2}(x_2\ y_3\ z_4) &= (1\ \ y_3\ z_4) \\ \frac{y_1}{b^2}(x_2\ y_3\ z_4) &= (x_2\ 1\ \ z_4) \\ \frac{z_1}{c^2}(x_2\ y_3\ z_4) &= (x_2\ y_3\ 1) \end{aligned}\right\} \quad \text{.........................(1).}$$

We shall have three other groups of equations of a similar character, viz.,

$$\left.\begin{array}{l}\frac{x_2}{a^2}(x_1\ y_3\ z_4)=(1\ \ y_3\ z_4)\\ \frac{y_2}{b^2}(x_1\ y_3\ z_4)=(x_1\ 1\ \ z_4)\\ \frac{z_2}{c^2}(x_1\ y_3\ z_4)=(x_1\ y_3\ 1)\end{array}\right\},\quad \left.\begin{array}{l}\frac{x_3}{a^2}(x_1\ y_2\ z_4)=(1\ \ y_2\ z_4)\\ \frac{y_3}{b^2}(x_1\ y_2\ z_4)=(x_1\ 1\ \ z_4)\\ \frac{z_3}{c^2}(x_1\ y_2\ z_4)=(x_1\ y_2\ 1)\end{array}\right\}\ldots(2,3),$$

$$\left.\begin{array}{l}\frac{x_4}{a^2}(x_1\ y_2\ z_3)=(1\ \ y_2\ z_3)\\ \frac{y_4}{b^2}(x_1\ y_2\ z_3)=(x_1\ 1\ \ z_3)\\ \frac{z_4}{c^2}(x_1\ y_2\ z_3)=(x_1\ y_2\ 1)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Multiplying the first equations of groups (1), (2), (3), (4), by x_1, $-x_2$, x_3, $-x_4$, and adding, we obtain

$$\frac{1}{a^2}\left\{x_1^2(x_2y_3z_4)-x_2^2(x_1y_3z_4)+x_3^2(x_1y_2z_4)-x_4^2(x_1y_2z_3)\right\}=(x_1y_2z_3 1).$$

Now $\xi=-\frac{(x_2y_3z_4)}{(x_1y_2z_3 1)}$, $\eta=\frac{(x_1y_3z_4)}{(x_1y_2z_3 1)}$, $\zeta=-\frac{(x_1y_2z_4)}{(x_1y_2z_3 1)}$, $\theta=\frac{(x_1y_2z_3)}{(x_1y_2z_3 1)}$.

Hence, as $\lambda_1=x_1$, $\mu_1=x_2$, $\nu_1=x_3$, $\pi_1=x_4$,

$$-a^2=\lambda_1^2\xi+\mu_1^2\eta+\nu_1^2\zeta+\pi_1^2\theta.$$

Similarly the other equations are established. The remainder follows easily.

9215. (S. TEBAY, B.A.)—The growth at any point of a blade of grass varies directly as its distance from the root. The respective heights of grass in three meadows, of 2, 3, and 5 acres, are 3, $3\frac{1}{2}$, and 4 inches. The grass in the first and second meadows is cut in 32 and 30 days, respectively. If 12 oxen consume the produce of the first meadow in 56 days, and 16 oxen consume the produce of the second meadow in 63 days, find when the grass in the third meadow must be cut so that 18 oxen may consume the produce in 80 days.

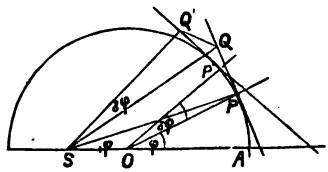
Solution by the PROPOSER.

If h be the height of the grass at first, and m the rate of growth at unity, the rate at the height $h+x$ is $m(h+x)=dx/dt$. Hence, if x vanishes with t, we have $h+x=he^{mt}$, which is the height of the grass at time t. Hence the consumptions in the three cases are $6e^{32m}$, $10\frac{1}{2}e^{30m}$, $20e^{mt}$. Now 12, 16, 18 oxen consume $6e^{32m}$, $10\frac{1}{2}e^{30m}$, $20e^{mt}$ in 56, 63, 80 days; therefore $\frac{1}{112}e^{32m}=\frac{1}{96}e^{30m}=\frac{1}{72}e^{mt}$. From the first equation we have $e^m=(\frac{7}{6})^{\frac{1}{2}}$; therefore $e^{32m}=(\frac{7}{6})^{16}=\frac{14}{9}e^{mt}=\frac{14}{9}(\frac{7}{6})^{\frac{1}{2}t}$,

$$\tfrac{1}{2}t=16-\frac{\log 14-\log 9}{\log 7-\log 6}=13{\cdot}13375,\ \text{and}\ t=26{\cdot}2675\ \text{days}.$$

8577. (B. HANUMANTA RAU, M.A.)—Prove that the arc of the pedal of a circle, of radius a, is equal to the arc of an ellipse ($e = \frac{4}{5}$), the origin being at a distance $\frac{1}{4}a$ from the centre of the circle.

Solution by Professor MATHEWS, M.A.; SARAH MARKS, B.Sc.; *and others.*



Let SP $= r$, $\angle$ POA $= \phi$; then QQ$'$ or $dS = r\,d\phi$, ultimately. Now we have

$$r^2 = \text{SO}^2 + \text{OP}^2 + 2\text{SO}\,.\,\text{OP}\cos\phi,$$

or, if SO $= \frac{1}{4}a$, OP $= a$,

$$r^2 = \left(\tfrac{17}{16} + \tfrac{1}{2}\cos\phi\right)a^2$$
$$= \left(\tfrac{25}{16} - \sin^2\tfrac{1}{2}\phi\right)a^2,$$

Hence, if $\phi = 2\theta$, we shall have

$$ds = \tfrac{5}{2}a\left(1 - \tfrac{16}{25}\sin^2\theta\right)d\theta, \quad s = \tfrac{5}{2}a\int\left(1 - e^2\sin^2\theta\right)d\theta,$$

where $e = \frac{4}{5}$, therefore, &c.

8855. (Professor MUKHOPÂDHYÂY, M.A., F.R.A.S.)—Prove that (1) the solution of the system $\dfrac{y}{x}\,.\,\dfrac{1+x^2}{1+y^2} = a$, $\dfrac{y^3}{x^3}\,.\,\dfrac{1+x^6}{1+y^6} = b^3$ is given by

$$x^2 = \frac{1}{\lambda}\,.\,\frac{\lambda - a}{a\lambda - 1}, \quad y^2 = \lambda\,\frac{\lambda - a}{a\lambda - 1},$$

where λ satisfies $\left(\dfrac{\lambda - a}{a\lambda - 1}\right)^3 = \dfrac{\lambda^3 - b^3}{b^3\lambda^3 - 1}$; and obtain (2) all the solutions by the transformation $\lambda + \lambda^{-1} = \mu$.

Solution by W. J. BARTON, M.A.; R. F. DAVIS, M.A.; *and others.*

Putting $y = \lambda x$, we have $\lambda\,\dfrac{1+x^2}{1+\lambda^2x^2} = a$, whence $x^2 = \dfrac{1}{\lambda}\,.\,\dfrac{\lambda - a}{a\lambda - 1}$ and therefore $y^2 = \lambda\,\dfrac{\lambda - a}{a\lambda - 1}$; hence, by substituting in the second equation, we get

$$\left(\frac{\lambda - a}{a\lambda - 1}\right)^3 = \frac{\lambda^3 - b^3}{b^3\lambda^3 - 1};$$

$$\therefore\ (b^3 - a^3)\left(\lambda^3 - \frac{1}{\lambda^3}\right) - 3a\,(b^3 - a)\left(\lambda^2 - \frac{1}{\lambda^2}\right) + 3a\,(ab^3 - 1)\left(\lambda - \frac{1}{\lambda}\right) = 0;$$

whence $\lambda - \lambda^{-1} = 0$, or $\lambda = \pm 1$, which give $x^2 + 1 = 0$, $y^2 + 1 = 0$, or, putting $\lambda + \lambda^{-1} = \mu$, $(b^3 - a^3)(\mu^2 - 1) - 3a\,(b^3 - a)\,\mu + 3a\,(ab^3 - 1) = 0$. Let μ_1, μ_2 be roots of this quadratic; then

$$\lambda^2 - \mu_1\lambda + 1 = 0, \quad \text{or} \quad \lambda^2 - \mu_2\lambda + 1 = 0,$$

the roots of which may be denoted by $\lambda_1, \frac{1}{\lambda_1}; \lambda_2, \frac{1}{\lambda_2}$. Substituting in values of x^2, y^2 above, we get $x = \pm x_1, \pm\frac{1}{x_1}; y = \pm y_1, \pm\frac{1}{y_1}$.

9140. (EMILE VIGARIÉ.)—Si R, R_1, R_2 désignent respectivement les rayons du cercle circonscrit du premier cercle de Lemoine (*triplicate ratio circle*) et le deuxième cercle de Lemoine (*cosine*), démontrer la relation

$$R^2 = 4R_1^2 - R_2^2.$$

Solution by Professors IGNACIO BEYENS; BORDAGE; *and others.*

D'après les valeurs de R_1 et R_2 qui sont déjà connus (voyez LIEBER, *Über die Gegenmittellinie und den Grebis'chen Punkt*) on a :

$$R_1 = \frac{R\,(b^2c^2 + a^2c^2 + a^2b^2)^{\frac{1}{2}}}{a^2+b^2+c^2}, \quad R_2 = \frac{abc}{a^2+b^2+c^2},$$

$$\therefore\ 4R_1^2 - R_2^2 = \frac{4R^2\,(a^2c^2 + b^2c^2 + a^2b^2) - a^2b^2c^2}{(a^2+b^2+c^2)^2}.$$

Mais, désignant Δ la surface du triangle, on a $\Delta = \frac{abc}{4R}$, et on déduira :

$$4R^2 = \frac{a^2b^2c^2}{4\Delta^2}, \text{ et } 4R_1^2 - R_2^2 = \frac{a^2b^2c^2\,(a^2c^2 + b^2c^2 + a^2b^2) - a^2b^2c^2 \,.\, 4\Delta^2}{(a^2+b^2+c^2)^2\,4\Delta^2}$$

$$= \frac{a^2b^2c^2\,(4a^2c^2 + 4b^2c^2 + 4a^2b^2 - 16\Delta^2)}{(a^2+b^2+c^2)^2\,16\Delta^2} = \frac{a^2b^2c^2\,(2a^2c^2 + 2b^2c^2 + 2a^2b^2 + a^4 + b^4 + c^4)}{(a^2+b^2+c^2)^2\,16\Delta^2}$$

$$= \frac{a^2b^2c^2\,(a^2+b^2+c^2)^2}{16\Delta^2\,(a^2+b^2+c^2)^2} = \frac{a^2b^2c^2}{16\Delta^2} = \left(\frac{abc}{4\Delta}\right)^2 = R^2.$$

9264. (Professor HUDSON, M.A.)—Prove that $y = \sqrt{2}\,(x-4a)$ is both a tangent and a normal to $27ay^2 = 4\,(x-2a)^3$.

Solution by R. F. DAVIS, M.A.; R. W. D. CHRISTIE, M.A.; *and others.*

The abscissæ of the points of intersection of the straight line and curve are given by the equation

$$27a\,(x-4a)^2 = 2\,(x-2a)^3,$$

$$\text{or}\quad 2x^3 - 39ax^2 + 240a^2x - 448a^3 = 0, \quad\text{or}\quad (x-8a)^2\,(2x-7a) = 0.$$

By examining the value of dy/dx at these points it will be found that the straight line is a tangent at the point $(8a, 4a\sqrt{2})$ and a normal at the point $(7a/2 - a/\sqrt{2})$.

[We may write the normal to $27ay^2 = 4(x-2a)^3$ as

$$y = mx - 2am - \frac{3a}{m} + \frac{2a}{m^3},$$

and the tangent (being the normal to $y^2 = 4ax$) as $y = mx - 2am - am^3$; and if these lines coincide, we have the equation $m^6 - 3m^2 + 2 = 0$, whereof the only real roots are $m = \pm\sqrt{2}$.]

9338. (A. Russell, B.A.)—Show that the solution of the partial differential equation

$$x^4 \frac{\partial^4 z}{\partial x^4} + 6x^3 \frac{\partial^3 z}{\partial x^3} + 7x^2 \frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial x} = a^2 \frac{\partial^2 z}{\partial y^2} - 2a^3 \frac{\partial z}{\partial y} + a^4 z$$

is
$$z = e^{ay} \int_0^\infty f\left(y \pm \frac{\theta^2}{2}\right) e^{\pm a(\log x)^2/2\theta^2} d\theta.$$

Solution by J. W. Sharpe, M.A.

Substitute $\zeta = \log x$; then $\frac{\partial^4 z}{\partial \zeta^4} = a^2 \left(\frac{\partial}{\partial y} - a\right)^2 z$; therefore the solution is the sum of the solutions of

$$\frac{\partial^2 z}{\partial \zeta^2} = a\left(\frac{\partial}{\partial y} - a\right) z, \text{ and } \frac{\partial^2 z}{\partial \zeta^2} = -a\left(\frac{\partial}{\partial y} - a\right) z.$$

Take the first, and put $z = Ae^{ay+hy}$, and $\frac{\partial^2}{\partial \zeta^2} = D^2$;

then $h = D^2/a$; therefore $z = e^{ay} . e^{y/a D^2} f(x)$,

f being arbitrary; therefore

$$z = e^{ay} \int_{-\infty}^{\infty} e^{-u^2} f\left(x + 2u\sqrt{\frac{y}{a}}\right) du.$$

Let $u^2 = \frac{ax^2}{2\theta^2}$, then we obtain for z the value

$$e^{ay} \int_0^\infty e^{-ax^2/2\theta^2} \left\{ f\left(x + \sqrt{(2y)}\,\frac{x}{\theta}\right) + f\left(x - \sqrt{(2y)}\,\frac{x}{\theta}\right) \right\} \frac{x\,d\theta}{\theta^2}.$$

The other solutions are obtained by changing the sign of a under the integral sign.

8752. (Professor Genese, M.A.)—If AL, BM, CN be perpendiculars from the vertices of a triangle ABC upon any straight line in its plane, then, three letters denoting an area, and signs being regarded, prove that

$$AMN + BNL + CLM = ABC.$$

Solution by the Proposer.

Let p, q, r be the perpendiculars from A, B, C on the line, then its equation in perpendicular coordinates is $ap\alpha + bq\beta + cr\gamma = 0$. The line

is therefore the line of action of the resultant R of forces ap, bq, cr along BC, CA, AB. Taking moments about A, $Rp = ap \cdot 2\Delta / a$, or $R = 2\Delta$. Let R make angles θ, ϕ, ψ with the sides. Resolving along the line,

$$ap\cos\theta + bq\cos\phi + cr\cos\psi = R \quad\text{or}\quad p.MN + q.NL + r.LM = 2\Delta,$$

whence the theorem. The line may be drawn through A and the theorem verified by Euc. I. 37, then easily extended.

9146. (R. Lachlan, M.A.)—If two circles (radii ρ, ρ') intersect in A and B, and any straight line cut them in the points (P, Q), (R, S) respectively, show that

$$(AP.BP.AQ.BQ)/\rho^2 = (AR.BR.AS.BS)/\rho'^2,$$
$$(AP.BP.AS.BS)/SP^2 = (AQ.BQ.AR.BR)/QR^2.$$

Solution by Professors Ignacio Beyens, Matz; *and others.*

(1) Soient PH, QH′ les hauteurs des triangles PAB, QAB, de même soient ST, RT′ les hauteurs de SAB, RAB : nous aurons :

$$PH.2\rho = AP.PB,\quad QH'.2\rho = AQ.QB,$$

d'où
$$PH.QH' = AP.PB.AQ.QB/4\rho^2.$$

De la même manière dans l'autre circonférence on a

$$ST.RT' = AR.BR.AS.BS/4\rho'^2;$$

mais les triangles semblables KRT′, KPH (K étant le point de rencontre de PS, AB) et KH′Q, KST nous donnent :

$$(\alpha)\quad \frac{PH}{RT'} = \frac{KP}{KR},\quad \frac{QH'}{ST} = \frac{KQ}{KS},\quad \text{et par suite}\quad \frac{PH.QH'}{RT'.ST} = \frac{KP.KQ}{KR.KS} = 1;$$

donc
$$AP.BP.AQ.BQ/\rho^2 = AR.BR.AS.BS/\rho'^2 \;............(1).$$

(2) Des relations $AP.BP = PH.2\rho$, $AS.BS = 2\rho'.TS$, on déduit : $AP.BP.AS.BS = 4\rho\rho'.PH.TS$, et d'une manière analogue on a : $AQ.BQ.AR.BR = 4\rho\rho'.RT'.QH'$, et par suite

$$\frac{AP.BP.AS.BS}{AQ.BQ.AR.BR} = \frac{PH.TS}{RT'.QH'};$$

mais des relations (α) on a :

$$\frac{PH.ST}{RT'.QH'} = \frac{KP}{KR}\cdot\frac{KS}{KQ} = \frac{KP^2}{KR^2} = \frac{KS^2}{KQ^2} = \frac{PS^2}{RQ^2},$$

car de $KP.KQ = KR.KS$ on obtiendra :

$$\frac{KP}{KR} = \frac{KS}{KQ} = \frac{PS}{QR},\quad\text{ou}\quad \frac{KP^2}{KR^2} = \frac{KS^2}{KQ^2} = \frac{PS^2}{QR},$$

et par suite la seconde rotation est demontrée.

[The results may be otherwise deduced from the following theorem :—
(α) If two circles, centres H, K, intersect in A and B, and if OP be the tangent from any point O on the former, drawn to the latter, then

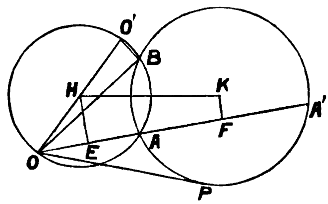
$$OA.OB = OP^2.OH/HK.$$

To prove this, let OA cut the other circle in A′, and let E, F be the mid-points of OA, AA′; and let O′ be opposite extremity of the diameter OH. Then the angles OO′B, EHK are equal, and therefore

$$\frac{EF}{HK} = \frac{OB}{OO'},$$

therefore $\frac{OB}{OA'} = \frac{OH}{HK}$,

whence the result follows.



[The theorem (α) is a particular case of the following general theorem :—(β) If through a fixed point O, a variable circle, with radius R and centre H, be drawn to intersect a circular curve of $(n+m)^{\text{th}}$ order, having n double foci $F_1, F_2, \ldots$, in the points $P_1, P_2 \ldots$; then the product of the distances $OP_1, OP_2, \ldots$, say (OP), varies as $R^m/(HF)$. This general theorem is easily proved, and may be regarded as an extension of CARNOT's theorem. The similar extension to the case of a circle cutting a non-circular curve has been given by LAGUERRE, *Comptes Rendus*, Vol. LX., pp. 71—73.]

9229, 9259, & 9301. (Professor SYLVESTER, F.R.S.)—(9229). Prove that the points of intersection of any given bicircular quartic by a transversal, will be foci of a hyper-cartesian capable of being drawn through four concyclic foci of the given quartic.

(9259). Prove that, if one set of four collinear points are the foci of a hyper-cartesian drawn through a second set of the same, the second set will be the collinear foci of a hyper-cartesian that can be drawn through the first set.

(9301). Prove that the points in which a pair of circles are cut by any transversal will be the collinear foci of a system of hyper-cartesians having double contact with one another at two points.

Solution by Professor NASH, M.A.

A hyper-cartesian is the inverse of a bicircular quartic with respect to a point on one of the focal circles. Hence the first two theorems can be at once derived from the general theorem that if F, G, H, K be concyclic points on a bicircular quartic of which A, B, C, D are concyclic foci, then a bicircular quartic may be described through A, B, C, D having F, G, H, K as foci. This may be proved as follows :—

Let $(x-\alpha)^2+(y-\beta)^2=\rho^2$ and $x^2+y^2=\rho'^2$ be the equations of the circles ABCD, FGHK, and $ax^2+2hxy+by^2+2gx+2fy+c=0$ that of the focal conic corresponding to ABCD; then the equation of the quartic is

$$CS^2-4S\left\{G(x-\alpha)+F(y-\beta)\right\}$$
$$+4A(x-\alpha)^2+8H(x-\alpha)(y-\beta)+4B(y-\beta)^2=0,$$

where $$S = x^2+y^2-\alpha^2-\beta^2+\rho^2 = x^2+y^2-t^2,$$

and $$A = bc-f^2,\quad B = ca-g^2,\ \&c.$$

At the intersections of the curve with the circle FGHK, $S = \rho'^2 - t^2 = t'^2$ suppose; hence, if these intersections lie on the conic,

$$4A(x-\alpha)^2 + 8H(x-\alpha)(y-\beta) + 4B(y-\beta)^2$$
$$-4t'^2 G(x-\alpha) - 4t'^2 F(y-\beta) + Ct'^4 = 0.$$

Forming the reciprocal coefficients, the equation of the bicircular quartic having this conic as focal conic and FGHK as focal circle is seen to be

$$cS'^2 + 2S't'^2(gx+fy) + t'^4(ax^2+2hxy+by^2) = 0,$$

where $$S' = (x-\alpha)^2 + (y-\beta)^2 - \rho^2 + t'^2;$$

hence at intersections with the circle ABCD, $S' = t'^2$, and these lie upon the conic $ax^2 + 2hxy + \&c. = 0$, *i.e.*, the quartic passes through ABCD.

(9301). Prof. CASEY has shown that two circles may be considered as a particular case of a bicircular quartic when the focal circle and conic have double contact, *i.e.*, when AB coincide and also CD. Therefore by what has already been proved, if F, G, H, K be the intersections of the pair of circles with a straight line or circle, a bicircular quartic can be described having FGHK as foci, and touching the focal conic at A and C.

9303. (Professor NEUBERG.)—Sur les côtés du triangle ABC, on construit trois triangles semblables BCD, CAE, ABF; démontrer que la somme $(DE)^2 + (EF)^2 + (FD)^2$ est minimum, lorsque les points D, E, F sont les sommets du premier triangle de Brocard.

Solution by the PROPOSER; R. F. DAVIS, M.A.; *and others.*

Soient λ, μ, ν les angles en B, C, D du triangle BCD que nous supposons tourné vers l'intérieur de ABC. On a:

$$BD = a\sin\mu/\sin\nu, \quad BF = c\sin\lambda/\sin\nu;$$

d'où, dans le triangle BDF:

$$(FD)^2 = \{a^2\sin^2\mu + c^2\sin^2\lambda - 2ac\sin\mu\sin\lambda\cos(B-\lambda-\mu)\}\operatorname{cosec}^2\nu,$$
$$\sigma \equiv (DE)^2 + (EF)^2 + (FD)^2$$
$$= \{(a^2+b^2+c^2)(\sin^2\mu+\sin^2\lambda) + 2\sin\mu\sin\lambda\,\Sigma\, ac\cos(B+\nu)\}\operatorname{cosec}^2\nu.$$

Or, si V est l'angle de BROCARD de ABC, et S l'aire ABC, on a:

$$a^2+b^2+c^2 = 4S\cot V, \quad \sin^2\mu + \sin^2\lambda - 2\sin\mu\sin\lambda\cos\nu = \sin^2\nu,$$
$$2\Sigma\, ac\cos(B+\nu) = \cos\nu\,\Sigma\, 2ac\cos B - \sin\nu\,\Sigma\, 2ac\sin B$$
$$= \cos\nu\,(a^2+b^2+c^2) - 12\,S\sin\nu;$$

donc $$\sigma = 4S\cot V + 12S\sin\lambda\sin\mu\cos(V+\nu)\operatorname{cosec}^2\nu\operatorname{cosec}V,$$
$$\sigma = a^2+b^2+c^2 + 12S\operatorname{cosec}V\,.\,DB\,.\,DC\cos(V+\nu)/a^2.$$

Construisons une droite CD′ rencontrant BD sous l'angle $BD'C = 90° - V$; le triangle BCD′ donnera $DD' = CD\cos(V+\nu)/\cos V$. Donc

$$\sigma = a^2+b^2+c^2 + 12S\cot V\,.\,DB\,.\,DD'/a^2.$$

Ainsi la différence $\sigma - (a^2+b^2+c^2)$ est proportionelle à la puissance du

point D par rapport au cercle du segment capable de l'angle $90° - V$ construit sur BC. Le centre de ce cercle est le sommet A_1 du premier triangle de BROCARD. La puissance DB . DD' a sa plus grande valeur négative lorsque D coïncide avec A_1; alors σ passe par un minimum.

Scolie.—Le lieu du point D tel que σ a une valeur constante est une circonférence ayant pour centre A_1. En particulier, lorsque l'angle $BCD = 90° - V$, on a : $\sigma = a^2 + b^2 + c^2$.

8868. (Professor SCHOUTE.)—If ABC and A'B'C' are two positions of the same triangle in space; if A'', B'', C'' are the centres of the segments AA', BB', CC', and if the planes through A'', B'', C'' respectively perpendicular to AA', BB', CC', intersect in P, the tetrahedrons PABC and PA'B'C' are not congruent, but symmetrical.

Solution by Professor G. J. LEGEBEKE.

Displacing first the triangle A'B'C' parallel to itself into the position AB_0C_0, A being the vertex of the triangle ABC corresponding to A', we may afterwards turn AB_0C_0 round an axis AO until it coincides with the triangle ABC. This axis AO is perpendicular to the lines B_0B and C_0C.

The point P considered as vertex of the tetrahedron PA'B'C' will share the movement of the base A'B'C' and first describe the line PP_0 equal in length and direction to A'A. If now the two tetrahedrons are congruent the rotational displacement of AB_0C_0 will bring P_0 to coincide with P, the vertex of the tetrahedron PABC; then the axis AO is perpendicular to the line P_0P. The line AO, being perpendicular to P_0P and consequently to AA', BB_0 and CC_0, is also perpendicular to the three lines AA', BB', CC', or, what is the same, the displacements of the vertices of the triangle ABC are parallel to the same plane. In this case, however, the planes which bisect and are perpendicular to the lines AA', BB', and CC', do not meet in one point. When, therefore, those three planes meet in one point, the tetrahedrons cannot be congruent, but must be symmetrical. When the displacements AA', BB', CC' are parallel to the same plane $BB'B_0$ or $CC'C_0$, the axis OA, being perpendicular to B_0B and C_0C, will in general be perpendicular to that plane. Now it is not difficult to prove that in this case the planes, which bisect and are perpendicular to the displacements, meet in one line. Therefore we project the figure on a plane parallel to the displacements or perpendicular to AO. Then, of course, the projections of the three positions of the triangle are congruent. Now the lines bisecting perpendicularly the lines joining the corresponding vertices of the projections of ABC and A'B'C' will go through the same point, and therefore the planes in question will meet in one line.

In particular, when the lines BB_0 and CC_0 are parallel, the axis AO need not be perpendicular to the parallel planes $BB'B_0$ and $CC'C_0$. In this case we find, by projecting the figure on a plane parallel to the displacements, that the bisecting planes meet in three parallel lines; for the projections of the triangles AB_0C_0 and A'B'C' are congruent, and that of ABC is symmetrical with that of AB_0C_0.

2437. (The late Rev. J. Blissard, M.A.)—Prove that

$$\frac{1}{1^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{5^2-x^2}+\ldots = \frac{\pi}{4x}\tan\frac{\pi x}{2}.$$

Solution by George Goldthorpe Storr, M.A.

We have $$\cos y = \left(1-\frac{4y^2}{\pi^2}\right)\left(1-\frac{4y^2}{3^2\pi^2}\right)\ldots,$$

hence $$\log\cos y = \log\frac{\pi^2-4y^2}{\pi^2}+\log\frac{3^2\pi^2-4y^2}{3^2\pi^2}+\&c.$$

Differentiating, we have $$\tan y = \frac{8y}{\pi^2-4y}+\frac{8y}{3^2\pi^2-4y^2}+\&c.,$$

and, putting $y = \frac{1}{2}\pi x$, we obtain the result given in the question.

8818. (Professor Mukhopâdhyây, B.A., F.R.S.E.)—Show that, (1) the equation of the directrix of the conic which is described having the origin for focus and osculates $b^2x^2+a^2y^2 = a^2b^2$ at the point ϕ, is

$$(a^{-2}-b^{-2})\,(ax\cos^3\phi - by\sin^3\phi) = 1;$$

(2) the envelope of this for different values of ϕ is the quartic

$$b^2x^{-2}+a^2y^{-2} = (ab^{-1}-ba^{-1})^2,$$

which curve is also the reciprocal polar of the evolute of the conic $a^2x^2+b^2y^2 = a^2b^2$ with respect to a circle whose radius is a mean proportional between the axes of the ellipse.

Solution by Professor Wolstenholme, M.A., Sc.D.

1. The equation of any conic having its focus at the origin is $x^2+y^2 = (\mathrm{A}x+\mathrm{B}y+\mathrm{C})^2$; and, if this osculate the conic $x^2/a^2+y^2/b^2=1$ at the point $(a\cos\phi,\, b\sin\phi)$, and we denote $(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}$ by r, the equation $r = \mathrm{A}a\cos\phi+\mathrm{B}b\sin\phi+\mathrm{C}$ must have three roots ϕ; or we may differentiate it twice with respect to ϕ. This operation gives for A, B the two equations $(a^2-b^2)\sin\phi\cos\phi\,.\,r^{-1} = \mathrm{A}a\sin\phi-\mathrm{B}b\cos\phi,$

$$(a^2-b^2)(a^2\cos^4\phi-b^2\sin^4\phi)\,r^{-3} = \mathrm{A}a\sin\phi+\mathrm{B}b\cos\phi;$$

whence $$\mathrm{A}a/(a^2-b^2)$$

$$= \sin\phi\sin\phi\cos\phi\,.\,r^{-1}+\cos\phi\,(a^2\cos^4\phi-b^2\sin^4\phi)\,r^{-3} = a^2\cos^3\phi\,.\,r^{-3};$$

and similarly $$\mathrm{B}b/(b^2-a^2) = b^2\sin^3\phi\,.\,r^{-3},$$

whence $$\mathrm{C}' = r\,(a^2-b^2)(a^2\cos^4\phi-b^2\sin^4\phi)\,r^{-3}$$

$$\equiv \left\{(a^2\cos^2\phi+b^2\sin^2\phi)^2-(a^2-b^2)(a^2\cos^4\phi-b^2\sin^4\phi)\right\}r^{-3}\equiv a^2b^2r^{-3}.$$

Hence $$\mathrm{A}:\mathrm{B}:\mathrm{C} = a\cos^3\phi : -b\sin^3\phi : a^2b^2/(a^2-b^2);$$

and the equation of the directrix $(\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0)$ is as stated.

2. In the envelope of the directrix, we have, on differentiating,

$$ax\cos\phi + by\sin\phi = 0,$$

or $$\frac{ax}{\sin\phi} = \frac{by}{-\cos\phi} = \frac{ax\cos^3\phi - by\sin^3\phi}{\sin\phi\cos\phi\,(\cos^2\phi+\sin^2\phi)} = \frac{a^2b^2}{b^2-a^2}\Big/(\sin\phi\cos\phi),$$

or $$\cos\phi = \frac{ab}{b^2-a^2}\,\frac{b}{x},\quad \sin\phi = \frac{ab}{a^2-b^2}\,\frac{a}{y},$$

whence the equation of the envelope is

$$\frac{b^2}{x^2}+\frac{a^2}{y^2} = \left(\frac{a^2-b^2}{ab}\right)^2 \equiv \left(\frac{a}{b}-\frac{b}{a}\right)^2.$$

The reciprocal polar of the evolute of the conic $x^2/a'^2+y^2/b'^2 = 1$, with respect to a circle $x^2+y^2 = k^2$ is the locus of the pole of the normal $\frac{a'x}{\cos\phi}-\frac{b'y}{\sin\phi} = a'^2-b'^2$ with respect to this circle; and if (XY) be its pole,

$$\frac{X}{k^2} = \frac{a'}{(a'^2-b'^2)\cos\phi},\quad \frac{Y}{k^2} = \frac{b'}{(b'^2-a'^2)\sin\phi},$$

and these equations coincide with those for the point of contact of the directrix with its envelope if

$$\frac{k^2a'}{a'^2-b'^2} = \frac{ab^2}{b^2-a^2},\quad \frac{k^2b'}{b'^2-a'^2} = \frac{a^2b}{a^2-b^2};$$

whence $aa' = bb' = k^2$. Thus the envelope of the directrix is the reciprocal polar of the evolute of the conic $x^2/a'^2+y^2/b'^2 = 1$ with respect to the circle $x^2+y^2 = k^2$ provided $aa' = bb' = k^2$, and one case is when $k^2 = ab$, $a' = b$, $b' = a$.

[If the osculating conic in this Question osculate in P and cut the ellipse again Q, the equation of PQ will be

$$\frac{x}{a\cos\phi}\left\{a^2\sin^2\phi\,(1+\sin^2\phi)+b^2\cos^4\phi\right\}$$
$$-\frac{y}{b\sin\phi}\left\{a^2\sin^4\phi+b^2\cos^2\phi\,(1+\cos^2\phi)\right\} = a^2\sin^2\phi-b^2\cos^2\phi,$$

the envelope of which it would be interesting to find. The equation of the conic having its focus at the origin and osculating the ellipse $x^2/a^2+y^2/b^2 = 1$ at the point $(a\cos\theta,\, b\sin\theta)$ is given as Question 1218 in WOLSTENHOLME'S *Book of Problems*.]

2396, 6931 & 8935. (W. S. B. WOOLHOUSE, F.R.A.S.)—Let ABCD be any convex quadrilateral, having the diagonals AC, BD intersecting in E; and let ρ, ρ' denote the ratios 2AE.EC : AC^2, 2BE.ED : BD^2 respectively. Then, if five points be taken at random on the surface of the quadrilateral, prove that the probabilities (1) that the five random points will be the apices of a convex pentagon, will be $\frac{1}{36}(11+5\rho\rho')$; (2) that the pentagon will have one, and one only, point reentrant, will be $\frac{5}{9}$; (3) that it will have two reentrant points, will be $\frac{5}{36}(1-\rho\rho')$.

Solution by the Proposer.

This question was designed as an exercise on my general theorem for all convex surfaces, an investigation of which is given as a solution to Quest. 2471 (Vol. viii., p. 100), and of which theorem the following brief extract contains all that relates to the question about to be discussed.

Theorem.—Let a given plane surface having a convex boundary of *any form whatever* be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; also let h, k denote the radii of gyration round the axes, M the total area, and

$$C = \tfrac{1}{2}\int x^2\,dx\,\frac{y^3+y'^3}{M} + 3\int x\,dx\,\frac{y\int y^2\,dx + y'\int y'^2\,dx}{M}.$$

Then, if five points be taken at random on the surface, the probabilities

$$\left.\begin{aligned} &\text{of a convex pentagon} = 1-\frac{10C}{M^2}+\frac{5h^2k^2}{M^2}\\ &\text{of one reentrant point} = \frac{10C}{M^2}-\frac{20h^2k^2}{M^2}\\ &\text{of two reentrant points} = \frac{15h^2k^2}{M} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

One object in giving this extract from the general theorem is to effect an improvement in the formula for determining the value of the subsidiary quantity C, and to conveniently adapt the same to polar coordinates. We have, according to the above,

$$MC = \tfrac{1}{2}\int x^2y^3\,dx + 3\int xy\,dx\int y^2\,dx,$$

in which the integration is to be carried round the entire boundary. For the purpose of modifying this formula, we are of course at liberty either to retain or reject any function which vanishes between limits. The function $\int xy\,dx$ is one of this kind, and therefore between limits we have

$$0 = \left(\int xy\,dx\right)\left(\int y^2dx\right) = \int xy\,dx\int y^2dx + \int y^2dx\int xy\,dx.$$

Subtracting three times this, the expression for MC becomes

$$\tfrac{1}{2}\int x^2y^3\,dx - 3\int y^2dx\int xy\,dx.$$

Deducting $0 = \frac{1}{6}x^3y^3 = \frac{1}{6}\int(x^2y^3dx + x^3y^2\,dy) = \frac{1}{2}\int x^2y^3dx + \frac{1}{2}\int d\,(xy^2)\,x^2y$,

the expression for MC is reduced to

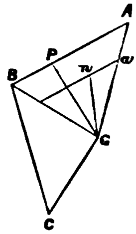
$$-3\int y^2dx\int xy\,dx - \tfrac{1}{2}\int d\,(xy^2)\,x^2y.$$

Lastly, we have $0 = xy^2\int xy\,dx = \int x^2y^3\,dx + \int d\,(xy^2)\int xy\,dx$,

by the addition of which we deduce

$$\begin{aligned} MC &= \int x^2y^3\,dx - 3\int y^2dx\int xy\,dx - \tfrac{1}{2}\int d\,(xy^2)\,x^2y + \int d\,(xy^2)\int xy\,dx\\ &= \int\left\{y^2dx - \tfrac{1}{2}d\,(xy^2)\right\}\left(x^2y - 3\int xy\,dx\right)\\ &= \tfrac{3}{2}\int (y^2dx - xy\,dy)\int(x^2dy - xy\,dx)\\ &= \tfrac{3}{2}\int y^3d\,\frac{x}{y}\int x^3d\,\frac{y}{x} = \tfrac{3}{2}\int R^3\,d\theta\,\sin\theta\int R^3\,d\theta\cos\theta \quad\ldots\ldots\ldots\ldots\ldots(2), \end{aligned}$$

the algebraic sign being necessarily reversed in the last expression for polar coordinates because x decreases when θ increases. The result just obtained is both elegant and symmetrical. Moreover, by changing θ into $\theta+\alpha$ it might now be easily shown that the value will be precisely the same for all rectangular axes of coordinates passing through the centre of gravity of the surface. This fact presents us with a most important advantage, as we are thereby relieved from all difficulty as to the former necessity of using the principal axes.



We now proceed to apply the last formula (2) to the case of the quadrilateral. In the annexed diagram G is the centre of gravity of the quadrilateral, and is taken as the origin of coordinates; GAB $= \Delta_1$, GBC $= \Delta_2$, &c., are the component triangles, P is a variable point xy in the side AB of the boundary, the integration being taken from A; GP = R, the angle AGP $= \theta$, $m =$ the sectorial area AGP, which is a triangle having its centre $\bar{x}\bar{y}$ at n, G$n = \bar{r}$, the angle aG$n = \bar{\theta}$, and $\bar{m} =$ the sectorial area aGn described by n. Then an is parallel to AP, making G$a = \frac{2}{3}$GA; also $\bar{m} = \frac{2}{9}m$. The coordinates of the centre of gravity of the sectorial element $dm = \frac{1}{2}R^2 d\theta$ being $\frac{2}{3}R\cos\theta$ and $\frac{2}{3}R\sin\theta$, we have

$$m\bar{x} = \tfrac{1}{3}\int R^3\,d\theta\cos\theta, \quad m\bar{y} = \tfrac{1}{3}\int R^3\,d\theta\sin\theta;$$

hence the first part of

$$\begin{aligned}\tfrac{3}{2}\mathrm{MC} &= \int d\,(3m\bar{y})(3m\bar{x}) = 9\int(m\,d\bar{y}+\bar{y}\,dm)\,m\bar{x} = 9\int m^2\bar{x}\,d\bar{y} + 9\int(m\,dm)\,\bar{x}\,\bar{y} \\ &= 9\int m^2\bar{x}\,d\bar{y} + \tfrac{9}{2}m^2\bar{x}\bar{y} - \tfrac{9}{2}\int m^2(\bar{x}\,d\bar{y}+\bar{y}\,d\bar{x}) \\ &= \tfrac{9}{2}\int m^2(\bar{x}\,d\bar{y}-\bar{y}\,d\bar{x}) + \tfrac{9}{2}m^2\bar{x}\bar{y} = \tfrac{9}{2}\int m^2\bar{r}^2\,d\theta + \tfrac{9}{2}m^2\bar{x}\bar{y} \\ &= 9\int m^2\,d\bar{m} + \tfrac{9}{2}m^2\,\bar{x}\bar{y} = 2\int m^2 dm + \tfrac{9}{2}m^2\,\bar{x}\bar{y} = \tfrac{2}{3}m^3 + \tfrac{9}{2}m^2\,\bar{x}\bar{y} \quad \ldots\ldots\ldots(3).\end{aligned}$$

When P has reached the corner B and the first component triangle is completed, then $m = \Delta_1$, $\bar{x} = \frac{1}{3}(x_1+x_2)$, $\bar{y} = \frac{1}{3}(y_1+y_2)$; and if we make

$$X_1 = \Delta_1(x_1+x_2), \quad X_2 = \Delta_2(x_2+x_3), \ \text{\&c.},$$
$$Y_1 = \Delta_1(y_1+y_2), \quad Y_2 = \Delta_2(y_2+y_3), \ \text{\&c.},$$

the first part of $\frac{3}{2}$MC will be $\frac{2}{3}\Delta_1^3 + \frac{1}{2}X_1Y_1$; and we shall have

$$X_1+X_2+X_3+X_4 = 0, \quad Y_1+Y_2+Y_3+Y_4 = 0.$$

In ascertaining the integration with respect to the second component triangle when P passes from B to C, it should be observed that in the formula $\quad \frac{3}{2}\mathrm{MC} = \int R^3\,d\theta\sin\theta\int R^3\,d\theta\cos\theta = \int d\,(3m\bar{y})(3m\bar{x})$,

the second factor $3m\bar{x}$ represents the integral $\int R^3\,d\theta\cos\theta$, and that it is indispensable that this integral shall always be counted from the same epoch. From B, its value having there become X_1, it must now be regarded as $3m\bar{x}+X_1$; and in the third and fourth parts of the integration it must, in like manner, be taken as $3m\bar{x}+X_2+X_1$ and $3m\bar{x}+X_3+X_2+X_1$ respectively, the quantities X_1, X_2, X_3, X_4 operating as constants. Thus the second, third, and fourth portions of the integral are found to be

$$\tfrac{2}{3}\Delta_2^3 + (\tfrac{1}{2}X_2+X_1)\,Y_2, \quad \tfrac{2}{3}\Delta_3^3 + (\tfrac{1}{2}X_3+X_2+X_1)\,Y_3 = \tfrac{2}{3}\Delta_3^3 - (\tfrac{1}{2}X_3+X_4)\,Y_3,$$
$$\tfrac{2}{3}\Delta_4^3 + (\tfrac{1}{2}X_4+X_3+X_2+X_1)\,Y_4 = \tfrac{2}{3}\Delta_4^3 - \tfrac{1}{2}X_4Y_4.$$

By collecting the four sections of the integral we obtain, as its complete value, $\frac{3}{2}\mathrm{MC} = \frac{2}{3}\Sigma(\Delta^3) + \frac{1}{2}X_1Y_1 + (\frac{1}{2}X_2 + X_1)\,Y_2 - (\frac{1}{2}X_3 + X_4)\,Y_3 - \frac{1}{2}X_4Y_4$.
Adding the equality

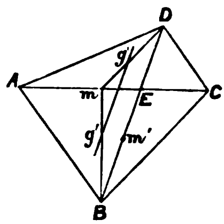
$$0 = -\tfrac{1}{2}(X_1 + X_2)(Y_1 + Y_2) + \tfrac{1}{2}(X_3 + X_4)(Y_3 + Y_4),$$

it reduces to

$$\tfrac{3}{2}\mathrm{MC} = \tfrac{2}{3}\Sigma(\Delta^3) + \tfrac{1}{2}(X_1Y_2 - X_2Y_1) + \tfrac{1}{2}(X_3Y_4 - X_4Y_3)$$
$$= \tfrac{2}{3}\Sigma(\Delta^3) + 4\,(\Delta_1\Delta_2 \,.\, \Delta''_{1,2} + \Delta_3\Delta_4 \,.\, \Delta''_{3,4}) \quad \text{.............}(4),$$

in which $\Delta''_{1,2}$ denotes the area of a triangle formed by the centre G and the middle points of the sides AB, BC; and $\Delta''_{3,4}$ the same with respect to the other two sides CD, DA.

We have yet to determine, from the geometry of the quadrilateral, the values of the several triangular areas involved in the last formula. Furthermore, we have afterwards to ascertain the value of h^2k^2 where h and k specially relate to the principal axes.



Let ABCD be the quadrilateral with the diagonals AC, BD intersecting in E; m the middle point of AC and m' that of BD; g the centre of gravity of the triangle CDA, and g' that of ABC. Put A$m = q$, mE $= v$, B$m' = q'$, m'E $= v'$, and the angle AEB $=$ E. Then, with AC and BD as oblique axes of coordinates, the line through g, g' is $x = -\frac{2}{3}v$. Similarly, the line through the centres of gravity of the triangles DAB, BCD is $y = -\frac{2}{3}v'$. The centre of gravity (G) of the quadrilateral is therefore $x = -\frac{2}{3}v$, $y = -\frac{2}{3}v'$. With this centre as the origin, the coordinates of the four corners A, B, C, D are respectively $x_1 = -(q + \frac{1}{3}v)$, $y_1 = \frac{2}{3}v'$; $x_2 = \frac{2}{3}v$, $y_2 = -(q' + \frac{1}{3}v')$; $x_3 = q - \frac{1}{3}v$, $y_3 = \frac{2}{3}v'$; $x_4 = \frac{2}{3}v$, $y_4 = q' - \frac{1}{3}v'$. The axes of coordinates are now respectively parallel to the diagonals AC, BD. To change the axis of y into rectangular axes, for x put $x + y\cos\mathrm{E}$, and for y put $y\sin\mathrm{E}$. Thus we get, for the rectangular coordinates of the four points, the following values:—

$$x_1 = -(q + \tfrac{1}{3}v) + \tfrac{2}{3}v'\cos\mathrm{E}, \qquad y_1 = \tfrac{2}{3}v'\sin\mathrm{E};$$
$$x_2 = \tfrac{2}{3}v - (q' + \tfrac{1}{3}v')\cos\mathrm{E}, \qquad y_2 = -(q' + \tfrac{1}{3}v')\sin\mathrm{E};$$
$$x_3 = (q - \tfrac{1}{3}v) + \tfrac{2}{3}v'\cos\mathrm{E}, \qquad y_3 = \tfrac{2}{3}v'\sin\mathrm{E};$$
$$x_4 = \tfrac{2}{3}v + (q' - \tfrac{1}{3}v')\cos\mathrm{E}, \qquad y_4 = (q' - \tfrac{1}{3}v')\sin\mathrm{E}.$$

The values of $2\Delta_1$, $2\Delta_2$, $2\Delta_3$, $2\Delta_4$ aro to be had from the expressions $x_1y_2 - x_2y_1$, $x_2y_3 - x_3y_2$, $x_3y_4 - x_4y_3$, $x_4y_1 - x_1y_4$ respectively, and are hence found to be

$$2\Delta_1 = \left\{(q + \tfrac{1}{3}v)(q' + \tfrac{1}{3}v') - \tfrac{4}{9}vv'\right\}\sin\mathrm{E},$$
$$2\Delta_2 = \left\{(q - \tfrac{1}{3}v)(q' + \tfrac{1}{3}v') + \tfrac{4}{9}vv'\right\}\sin\mathrm{E},$$
$$2\Delta_3 = \left\{(q - \tfrac{1}{3}v)(q' - \tfrac{1}{3}v') - \tfrac{4}{9}vv'\right\}\sin\mathrm{E},$$
$$2\Delta_4 = \left\{(q + \tfrac{1}{3}v)(q' - \tfrac{1}{3}v') + \tfrac{4}{9}vv'\right\}\sin\mathrm{E}.$$

These values also give

$$\Delta_1 + \Delta_2 = q\,(q' + \tfrac{1}{3}v')\sin\mathrm{E}, \qquad \Delta_3 + \Delta_4 = q\,(q' - \tfrac{1}{3}v')\sin\mathrm{E},$$
$$\Delta_1 + \Delta_3 = (qq' - \tfrac{1}{3}vv')\sin\mathrm{E}, \qquad \Delta_2 + \Delta_4 = (qq' + \tfrac{1}{3}vv')\sin\mathrm{E};$$
$$\mathrm{M} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 2qq'\sin\mathrm{E} \quad \text{................}(5).$$

Again, the coordinates of the mid-points of AB, BC, CD, DA are

$$x_1'' = -\tfrac{1}{2}\left\{(q-\tfrac{1}{3}v)+(q'-\tfrac{1}{3}v')\cos E\right\}, \quad y_1'' = -\tfrac{1}{2}(q'-\tfrac{1}{3}v')\sin E;$$
$$x_2'' = \tfrac{1}{2}\left\{(q+\tfrac{1}{3}v)-(q'-\tfrac{1}{3}v')\cos E\right\}, \quad y_2'' = -\tfrac{1}{2}(q'-\tfrac{1}{3}v')\sin E;$$
$$x_3'' = \tfrac{1}{2}\left\{(q+\tfrac{1}{3}v)+(q'+\tfrac{1}{3}v')\cos E\right\}, \quad y_3'' = \tfrac{1}{2}(q'+\tfrac{1}{3}v')\sin E;$$
$$x_4'' = -\tfrac{1}{2}\left\{(q-\tfrac{1}{3}v)-(q'+\tfrac{1}{3}v')\cos E\right\}, \quad y_4'' = \tfrac{1}{2}(q'+\tfrac{1}{3}v')\sin E.$$

From these, we find

$$4\Delta_{1,2}'' = 2(x_1''y_2''-x_2''y_1'') \quad\text{and}\quad 4\Delta_{3,4}'' = 2(x_3''y_4''-x_4''y_3'')$$
$$4\Delta_{1,2}'' = q(q'-\tfrac{1}{3}v')\sin E = \Delta_3+\Delta_4,$$
$$4\Delta_{3,4}'' = q(q'+\tfrac{1}{3}v')\sin E = \Delta_1+\Delta_2 \quad\text{..................(6).}$$

Hence, by substitution in (4), we find

$$\tfrac{3}{2}MC = \tfrac{2}{3}\Sigma(\Delta^3)+\Delta_1\Delta_2(\Delta_3+\Delta_4)+\Delta_3\Delta_4(\Delta_1+\Delta_2)$$
$$= \tfrac{2}{3}\Sigma(\Delta^3)+\Delta_1\Delta_3(\Delta_2+\Delta_4)+\Delta_2\Delta_4(\Delta_1+\Delta_3) \quad\text{............(7).}$$

We have also $\qquad 8\Sigma(\Delta^3) = 8(\Delta_1^3+\Delta_2^3+\Delta_3^3+\Delta_4^3)$

$$= \left\{(q+\tfrac{1}{3}v)(q'+\tfrac{1}{3}v')-\tfrac{4}{9}vv'\right\}^3\sin^3E+\left\{(q-\tfrac{1}{3}v)(q'+\tfrac{1}{3}v')+\tfrac{4}{9}vv'\right\}^3\sin^3E$$
$$+\left\{(q-\tfrac{1}{3}v)(q'-\tfrac{1}{3}v')-\tfrac{4}{9}vv'\right\}^3\sin^3E+\left\{(q+\tfrac{1}{3}v)(q'-\tfrac{1}{3}v')+\tfrac{4}{9}vv'\right\}^3\sin^3E$$
$$= 4qq'(q^2+\tfrac{1}{3}v^2)(q'^2+\tfrac{1}{3}v'^2)\sin^3E;$$

$$4\Delta_1\Delta_3(\Delta_2+\Delta_4)+4\Delta_2\Delta_4(\Delta_1+\Delta_3)$$
$$= \left\{(q+\tfrac{1}{3}v)(q'+\tfrac{1}{3}v')-\tfrac{4}{9}vv'\right\}\left\{(q-\tfrac{1}{3}v)(q'-\tfrac{1}{3}v')-\tfrac{4}{9}vv'\right\}(qq'+\tfrac{1}{3}vv')\sin^3E$$
$$+\left\{(q-\tfrac{1}{3}v)(q'+\tfrac{1}{3}v')+\tfrac{4}{9}vv'\right\}\left\{(q+\tfrac{1}{3}v)(q'-\tfrac{1}{3}v')+\tfrac{4}{9}vv'\right\}(qq'-\tfrac{1}{3}vv')\sin^3E$$
$$= 2qq'\left\{(q^2-\tfrac{1}{9}v^2)(q'^2-\tfrac{1}{9}v'^2)-\tfrac{16}{81}v^2v'^2\right\}\sin^3E,$$

whence
$$\tfrac{3}{2}MC = \tfrac{1}{3}qq'(q^2+\tfrac{1}{3}v^2)(q'^2+\tfrac{1}{3}v'^2)\sin^3E$$
$$+\tfrac{1}{2}qq'\left\{(q^2-\tfrac{1}{9}v^2)(q'^2-\tfrac{1}{9}v'^2)-\tfrac{16}{81}v^2v'^2\right\}\sin^3E$$
$$= \tfrac{5}{6}qq'\left\{q^2q'^2-\tfrac{1}{15}(q^2-v^2)(q'^2-v'^2)\right\}\sin^3E$$
$$= \frac{M^3}{9}\left(1-\frac{q^2-v^2}{4q^2}\cdot\frac{q'^2-v'^2}{4q'^2}\right) = \frac{M^3}{9}\left(1-\frac{\rho\rho'}{4}\right),$$
$$\frac{C}{M^2} = \frac{2}{27}\left(1-\frac{\rho\rho'}{4}\right) \quad\text{..............................(8).}$$

It now only remains to determine the radii of gyration round the principal axes of the quadrilateral. Suppose one of these axes to make an angle ϕ with the axis of x which has been taken parallel to the diagonal AC. The ordinate of any point xy when referred to this principal axis will have a value equal to $y\cos\phi - x\sin\phi$; and accordingly the new ordinates of the four corners will now be the following:—

$$y_1 = \tfrac{2}{3}v'\sin(E-\phi)+(q+\tfrac{1}{3}v)\sin\phi, \qquad y_3 = \tfrac{2}{3}v'\sin(E-\phi)-(q-\tfrac{1}{3}v)\sin\phi,$$
$$y_2 = -(q'+\tfrac{1}{3}v')\sin(E-\phi)-\tfrac{2}{3}v\sin\phi, \quad y_4 = (q'-\tfrac{1}{3}v')\sin(E-\phi)-\tfrac{2}{3}v\sin\phi;$$
$$y_1+y_3 = \tfrac{4}{3}v'\sin(E-\phi)+\tfrac{2}{3}v\sin\phi,$$
$$y_2+y_4 = -\tfrac{2}{3}v'\sin(E-\phi)-\tfrac{4}{3}v\sin\phi;$$

and hence, by the theorem given by me in Quest. 8922, we get

$$6h^2 = -(y_1+y_3)(y_2+y_4)-y_1y_3-y_2y_4$$
$$= (q^2+\tfrac{1}{3}v^2)\sin^2\phi+(q'^2+\tfrac{1}{3}v'^2)\sin^2(E-\phi)+\tfrac{1}{3}vv'\sin\phi\sin(E-\phi).$$

To abbreviate, let $\alpha = q^2+\frac{1}{3}v^2$, $\beta = q'^2+\frac{1}{3}v'^2$, $\gamma = \frac{4}{3}vv'$; then

$$12h^2 = \alpha(1-\cos 2\phi)+\beta\left\{1-\cos(2E-2\phi)\right\}+\gamma\left\{\cos(E-2\phi)-\cos E\right\}$$
$$= \alpha+\beta-\gamma\cos E-(\alpha+\beta\cos 2E-\gamma\cos E)\cos 2\phi$$
$$-(\beta\sin 2E-\gamma\sin E)\sin 2\phi.$$

To further abbreviate, denote this by

$$12h^2 = U-V\cos 2\phi-W\sin 2\phi;$$

then for the other principal axis, changing 2ϕ into $\pi+2\phi$, we get

$$12k^2 = U+V\cos 2\phi+W\sin 2\phi.$$

Therefore $6(h^2+k^2) = U$, $6(h^2-k^2) = -V\cos 2\phi-W\sin 2\phi$.

Now, when these relations refer to the principal axes of the quadrilateral, the value of h^2-k^2 must be either a maximum or a minimum as regards the variable angle 2ϕ. Hence, by differentiation,

$$0 = V\sin 2\phi-W\cos 2\phi.$$

Adding together the squares of the last two equations, we get

$$36(h^2-k^2)^2 = V^2+W^2,$$

which being subtracted from the square of $6(h^2+k^2) = U$, we obtain

$$144h^2k^2 = U^2-(V^2+W^2) = (4\alpha\beta-\gamma^2)\sin^2 E$$
$$= \left\{4(q^2+\tfrac{1}{3}v^2)(q'^2+\tfrac{1}{3}v'^2)-\tfrac{16}{9}v^2v'^2\right\}\sin^2 E$$
$$= \tfrac{4}{3}\left\{4q^2q'^2-(q^2-v^2)(q'^2-v'^2)\right\}\sin^2 E.$$

Therefore $$h^2k^2 = \frac{M^2}{108}\left(1-\frac{q^2-v^2}{2q^2}\cdot\frac{q'^2-v'^2}{2q'^2}\right) = \frac{M^2}{108}(1-\rho\rho') \quad\text{......(9).}$$

Hence, by the general theorem premised at the beginning of this solution, if five points be taken at random on the surface of the quadrilateral, the probability of their being the corners of a convex pentagon

$$= 1-\frac{10C}{M^4}+\frac{5h^2k^2}{M^2} = 1-\frac{20}{27}\left(1-\frac{\rho\rho'}{4}\right)+\frac{5}{108}(1-\rho\rho') = \frac{11+5\rho\rho'}{36}.$$

And the probability of

$$\text{one reentrant point} = \frac{10C}{M^2}-\frac{20h^2k^2}{M^2} = \frac{20}{27}\left(1-\frac{\rho\rho'}{4}\right)-\frac{5}{27}(1-\rho\rho') = \frac{5}{9},$$

$$\text{two reentrant points} = \frac{15h^2k^2}{M^2} = \frac{5}{36}(1-\rho\rho').$$

[That the probability of one reentrant point should be for all quadrilaterals the simple fraction $\frac{5}{9}$, is most remarkable.]

9293. (Elizabeth Blackwood.)—Find the number of permutations of n letters, taken k together, repetition being allowed, but no three consecutive letters being the same; and prove that, if this number be denoted by P_k,

$$P_{k+1}-P_k = (n^2-n)\frac{\alpha^k-\beta^k}{\alpha-\beta},$$

where α, β are the roots of the equation $x^2-(n-1)x-(n-1) = 0$.

Solution by Professor R. SWAMINATHA AIYAR, B.A.

Of the P_k permutatious taken k together, let those that do not begin with any specified letter, say a, be represented by p_k in number; of these p_k permutations those that begin with a single b are evidently p_{k-1} in number, and those that begin with two b's are p_{k-2}. We thus have

$$P_k = p_k + p_{k-1} + p_{k-2}, \quad \text{and} \quad p_k = (n-1)(p_{k-1} + p_{k-2}).$$

Observing that $p_1 = n-1$, we see that $p_1, p_2, p_3 \ldots$ are the successive coefficients in the development of

$$\frac{1}{1-(n-1)x-(n-1)x^2}, \quad \text{therefore} \quad p_{k+1} = \frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta},$$

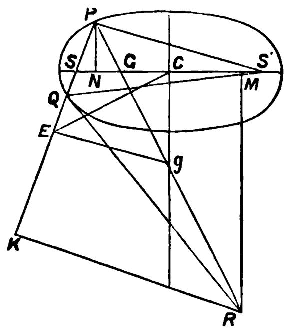
and
$$P_{k+1} - P_k = \frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta} - \frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta}.$$

This seems to be the correct result.

9378. (Rev. J. J. MILNE, M.A.)—PSQ is a focal chord of a conic. The normal at P (x_1, y_1) and the tangent at Q intersect in R. Show that the coordinates of R and the locus of R are respectively

$$\left(-x_1, \ -\frac{2a^2-b^2}{b^2}y_1\right), \quad \frac{x^2}{a^2} + \frac{b^2y^2}{(2a^2-b^2)^2} = 1.$$

Solution by C. E. WILLIAMS, M.A.; R. KNOWLES, B.A.; *and others.*



Let the normal PR meet the axes in G, g, and the diameter conjugate to CP meet PSQ in E; then PE = CA, and PEg is a right angle. Again, R is the centre of the escribed circle of the triangle PS′Q, whose perimeter = 4CA; hence, if RK be drawn perpendicular to PQ and parallel to Eg, we have

PK = semi-perimeter = 2CA;

therefore PE = EK, Pg = gR,

and CN = CM;

also RM : PN = GM : GN
= CN + CG : CN − CG
= $1+e^2 : 1-e^2$
= $2a^2-b^2 : b^2$.

[The equations to PSQ and the other focal chord PS′Q′, are

$$y(x_1-ae) = y_1(x-ae), \quad y(x_1+ae) = y_1(x+ae);$$

hence, combining these with the equation to the curve, we ought to get the tangent at P, and the chord QQ′, so that

$$\{y(x_1-ae)-y_1(x-ae)\}\{y(x_1+ae)-y_1(x+ae)\} + \lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right]$$
$$\equiv (Ax+By+C)\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right);$$

therefore QQ′ is, by comparing coefficients,

$$\frac{xx_1}{a^2}+\frac{(2a^2-b^2)\,yy_1}{b^4}+1=0,$$

and the pole of QQ′ is therefore $\left(-x_1,\ -\frac{2a^2-b^2}{b^2}y_1\right)$, which can be shown to lie on the normal at P, and is therefore the point R required.]

9337. (W. J. C. Sharp, M.A.)—If S_r denote $1^r+2^r\ldots+n^r$, prove that (1) $rS_{r-1}+\frac{r(r-1)}{1\,.\,2}S_{r-2}+\frac{r(r-1)(r-2)}{1\,.\,2\,.\,3}S_{r-3}+\ldots+S_0=(n+1)^r-1$; (2) deduce therefrom Fermat's Theorem; also (3) show that

$$S_r=(n+1)\left\{\frac{(n)^{(r)}}{r+1}+\frac{\Delta^{r-1}0^r}{(r-1)\,!}\,\frac{(n)^{(r-1)}}{r}+\frac{\Delta^{r-2}0^r}{(r-2)\,!}\,\frac{(n)^{(r-2)}}{r-1}+\&\text{c.}\right\},$$

where $(n)^{(r)}$ stands for $n(n-1)\ldots(n-r+1)$.

Solution by R. Knowles, B.A.; Professor Matz, M.A.; *and others.*

It is known that $\quad S_r=\frac{n^{r+1}}{r+1}+\tfrac{1}{2}rS_{r-1}-\frac{r(r-1)}{2\,.\,3}S_{r-2}+\&\text{c.},$

and hence we obtain $\quad S_0=n,\quad S_1=\tfrac{1}{2}n(n+1)$, &c.

In the series in the Question, putting $r=1, 2, 3$, &c., we have

$$S_0=n=(1+n)^1-1,$$

$$2S_1+S_0=n^2+2n=(1+n)^2-1,\quad 3S_2+3S_1+S_0=(1+n)^r-1;$$

therefore $\quad rS_{r-1}+\frac{r(r-1)}{1\,.\,2}S_{r-2}+\ldots S_0=(1+n)^1-1,$

therefore $\quad r\left\{S_{r-1}+\frac{(r-1)}{1\,.\,2}S_{r-2}+\&\text{c.}\right\}=(1+n)\left\{(1+n)^{r-1}-1\right\};$

hence, when r and $1+n$ are prime to each other, $(1+n)^{r-1}-1$ is divisible by r, which is Fermat's Theorem.

(3) In De Morgan's *Calculus*, p. 257, the formula

$$S_r=(n+1)\left\{\frac{\Delta 0^r}{2}\,.\,n+\frac{\Delta^2 0^r}{2\,.\,3}n(n-1)+\&\text{c.}\right\}$$

is given, its rth term being $\quad \frac{\Delta^r 0^r}{r\,!}\,.\,\frac{(n)^r}{r+1}=\frac{n^{(r)}}{r+1},$

and with this substitution we have formula (3).

[We may prove (1) thus, without assuming the value of S_r:—

$$S_r+rS_{r-1}+\frac{r(r-1)}{1.2}S_{r-2}+\ldots+S_0=\left\{n^r+r.n^{r-1}+\frac{r(r-1)}{1.2}n^{r-2}+\ldots+1\right\}$$

$$+\left\{(n-1)^r+r(n-1)^{r-1}+\frac{r(r-1)}{1.2}(n-1)^{r-2}+\ldots+1\right\}+\&\text{c.}$$

$$=(n+1)^r+n^r+(n-1)^r+\ldots+2^r=(n+1)^r+S_r-1^r;$$

and hence (1) holds. The slightly more general formula

$$rS_{r-1}+\frac{r(r-1)}{1.2}S_{r-2}+\ldots+S_0=(x+n)^r-x^r,$$

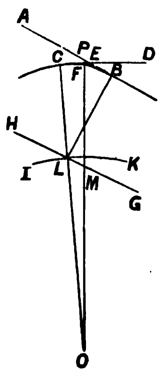
where $$S_r\equiv x^r+(x+1)^r+(x+2)^r+\ldots+(x+n-1)^r,$$

may be proved in the same way.

Thus, if r be a prime number, $(m+n)^r-m^r-n$ is divisible by r.]

2448. (J. S. Berriman, M.A.)—Let AEB, CED be two lines of railway, whereof AB is perfectly straight, and CD curved as far as F, the remainder being straight; then, if FE be 25 feet long, and the curve CF have a radius of 3000 feet, and the angle BED = 25° 26′; show that the distance from B to E, so that a curve BC may be struck with 1000 feet radius is 342·765 feet.

Solution by D. Biddle.



Let O be the centre of the curve CF, of radius 3000 ft. Draw the arc IK with the same centre, O, and radius 2000 ft., and, taking the point G, 1000 ft. from the line AB (produced), draw GH parallel to AB, cutting the arc IK in L. Then L is the centre of the required curve. In order to find the length of EB, produce OF to cut AB in P; then

$$OP = 3000+25\tan 25°\,26' = 3011{\cdot}8887025.$$

Again, $MP = 1000\sec 25°\,26' = 1107{\cdot}3147$,

and $OM = OP-MP = 1904{\cdot}5740025$.

Moreover, $OL : \sin LMO = OM : \sin MLO$;

whence $\angle MLO = 24°\,8'\,24{\cdot}5''$,

and $LOM = 1°\,17'\,35{\cdot}5''$; also $LM = 105{\cdot}1$.

Now, $BP = 1000\tan 25°\,26' - LM = 370{\cdot}4481$,

and $EB = BP-25\sec 25°\,26' = 342{\cdot}76523$ ft.,

the required distance.

9304. (Professor Schoute.)—Of a triangle ABC there is given the vertex A, the angle A, and the line of which BC is a part; find the loci of the remarkable points of the triangle ABC.

Solution by R. F. Davis, M.A.

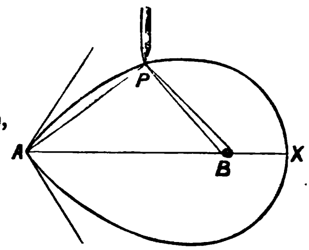
The locus of the orthocentre H is the perpendicular AD on BC, and that of the centroid G is a line parallel to BC trisecting AD. If OM be the perpendicular from the circumcentre on BC, $OM = R\cos A$; hence the locus of O is a hyperbola having A for focus, BC for directrix, and eccentricity $= \sec A$. Similarly the loci of I, J_1, J_2, J_3 the in- and ex-

centres are two hyperbolas having the same focus and directrix as the preceding one, and eccentricities cosec $\frac{1}{2}$A and sec $\frac{1}{2}$A respectively. If O′ be the image of O with respect to BC, and N the mid-point of AO′; then N is the Nine-points centre and describes a curve similar to the locus of O′, or of O, and is also a hyperbola.

9359. (J. O'Byrne Croke, M.A.)—Prove that the area of the simple Cartesian oval formed by guiding a pencil by a thread having one end attached to the tracing point and brought once tensely round a fixed pin of negligible section, the other being fastened to a second pin at a distance a from the former, and the whole length of the thread being $2a$, is

$$\tfrac{8}{9}a^2(2\pi-3\sqrt{3}).$$

Solution by H. Fortey, M.A.; D. Biddle; *and others.*



Let P be a point on the curve,

$$AP = r, \quad \text{and} \quad \angle PAB = \theta,$$

and area of curve $\doteq$ A; then

$$AB = a, \quad 2BP = 2a-r,$$

and $4BP^2 = 4(AB^2+AP^2-2AB.AP\cos\theta)$,

$$(2a-r)^2 = 4(a^2+r^2-2ar\cos\theta),$$

or $$3r = 4a(2\cos\theta-1);$$

therefore $$A = \tfrac{1}{2}\int r^2 d\theta$$

$$= \tfrac{8}{9}a^2\int(4\cos^2\theta-4\cos\theta+1)\,d\theta$$

$$= \tfrac{8}{9}a^2\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}(2\cos 2\theta-4\cos\theta+3)\,d\theta = \tfrac{8}{9}a^2(\sin 2\theta-4\sin\theta+3\theta)$$

$$= \tfrac{8}{9}a^2(2\pi-3\sqrt{3}).$$

[The curve as algebraically represented is a Cartesian oval; but we are concerned only with the inner loop, which is the only part of the curve that can be generated in the manner described.]

9250. (Major-General P. O'Connell.)—If s = the length of an arc of a circle, v = the versed sine of half the angle subtended by the arc, c = the chord of the arc; required a series for the value of s in terms of v and c.

Solution by R. W. D. Christie, M.A.; Sarah Marks, B.Sc.; *and others.*

We have $$v = (1-\cos\tfrac{1}{2}\theta), \quad c = 2r\sin\tfrac{1}{2}\theta, \quad s = r\theta;$$

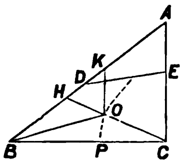
therefore $$\frac{c}{2vr} = \frac{\sin\frac{1}{2}\theta}{1-\cos\frac{1}{2}\theta} = \tan\tfrac{1}{4}\theta;$$

hence, by GREGORY's series, we get

$$\frac{S}{4r} = \tfrac{1}{2}\theta = \frac{c}{2vr} - \tfrac{1}{3}\left(\frac{c}{2vr}\right)^3 + \tfrac{1}{5}\left(\frac{c}{2vr}\right)^5 \ldots + \&c.$$

9367. (F. MORLEY, B.A.)—In the sides AB, AC of a triangle ABC, find points D, E, such that BD = DE = EC.

Solution by Prof. W. P. CASEY, M.A.; D. BIDDLE; *and others.*



Make AH = AC, and divide BC in P in the ratio of AH : HC, and let the arc PO be the locus of that ratio. Join BO and draw OK parallel to AC Then KH : HO = BO : OC = AK : OC, and therefore BO = AK. Make BD = BO and AE = KO, hence BO = BD = DE = EC.

[Take I the in-centre of the triangle — a figure is readily drawn or imagined—and P on BA, so that BP = CA; draw PQR parallel to AI, cutting the circle BIC in Q, R; then if BQ, CQ meet AC, AB in E, D, these will be points required. Another solution will be obtained by using P in place of Q, unless $\angle A = 60^\circ$. Similarly, if the lines are drawn in different directions AB, CA, or BA, AC. We thus have a construction for a triangle when they are given $a+b$, $a+c$, and LA. The problem has been long since solved in our columns, in its *present* form, and a generalized fôrm of it is solved, under part (2) of the EDITOR's Question 7675, on p. 64 of our Vol. XLII.]

8331. (H. G. DAWSON, B.A.)—Show that the solution of

$$\frac{x-y}{y^n} + \frac{x-z}{z^n} = ax, \quad \frac{y-x}{x^n} + \frac{y-z}{z^n} = by, \quad \frac{z-x}{x^n} + \frac{z-y}{y^n} = cz \ldots\ldots(1,\ 2,\ 3),$$

depends on the solution of

$$a(\rho-a)^{n-1} + b(\rho-b)^{n-1} + c(\rho-c)^{n-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Solutions by Professor AIYAR, B.A.; *the* PROPOSER; *and others.*

Divide (1) by x^n, (2) by y^n, (3) by z^n, and add; then

$$\frac{a}{x^{n-1}} + \frac{b}{y^{n-1}} + \frac{c}{z^{n-1}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Again, $$\frac{a-b}{b-c} = \left(\frac{1}{y} - \frac{1}{x}\right) \bigg/ \left(\frac{1}{z} - \frac{1}{y}\right),$$

whence we have $\dfrac{1}{x} : \dfrac{1}{y} : \dfrac{1}{z} = (\rho-a) : (\rho-b) : (\rho-c)$,

where ρ is a quantity that has to be determined. Substituting these ratios in (5), we have (4).

[*Otherwise:* (1) − (2) gives $(x-y)(x^{-n}+y^{-n}+z^{-n}) = ax-by$ ...... (6). Treating (2), (3), and (3), (1) similarly, we have

$$\frac{ax-by}{x-y} = \frac{by-cz}{y-z} = \frac{cz-ax}{z-x} = \sigma \text{ say},$$

therefore $$(a-\rho)\,x = (b-\rho)\,y = (c-\rho)\,z = \sigma,$$

therefore $$x = \frac{\sigma}{a-\rho},\quad y = \frac{\sigma}{b-\rho},\quad z = \frac{\sigma}{c-\rho};$$

substituting these expressions in (1), (2), and eliminating σ^n, we get (4).

From (6), we see that $\sigma^n = (a-\rho)^n + (b-\rho)^n + (c-\rho)^n$.]

8315. (Professor Booth, M.A.)—If

$$\tan^m(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi) = \tan^n(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

show that $$m \tan^{-1}\left(\frac{\sin\psi}{(-1)^{\frac{1}{2}}}\right) = n \tan^{-1}\left(\frac{\sin\phi}{(-1)^{\frac{1}{2}}}\right).$$

Solution by Professors Mahendra Nath Ray, LL.B., *and* Aiyar, B.A.

$$\tan^m(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi) = \left\{\tan^2(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi)\right\}^{\frac{1}{2}m} = \left(\frac{1+\sin\psi}{1-\sin\psi}\right)^{\frac{1}{2}m}.$$

Therefore $$\log\tan^m(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi)$$

$$= \tfrac{1}{2}m \,.\, 2\,(\sin\psi - \tfrac{1}{3}\sin^3\psi + \tfrac{1}{5}\sin^5\psi - \tfrac{1}{7}\sin^7\psi \ldots) = mi\tan^{-1}\left(\frac{\sin\psi}{(-1)^{\frac{1}{2}}}\right).$$

Similarly, $$\log\tan^n(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) = in\tan^{-1}\left(\frac{\sin\theta}{(-1)^{\frac{1}{2}}}\right).$$

9352. (Professor Hudson, M.A.)—Prove that

$$(\tan 7\tfrac{1}{2}^\circ + \tan 37\tfrac{1}{2}^\circ + \tan 67\tfrac{1}{2}^\circ)(\tan 22\tfrac{1}{2}^\circ + \tan 52\tfrac{1}{2}^\circ + \tan 82\tfrac{1}{2}^\circ) = 17 + 8\sqrt{3}.$$

Solution by R. W. D. Christie, M.A.; G. G. Storr, M.A.; *and others.*

We have $$\tan 7\tfrac{1}{2}^\circ = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1);$$

$$\tan 37\tfrac{1}{2}^\circ = (\sqrt{3}-\sqrt{2})(\sqrt{2}+1);\quad \tan 67\tfrac{1}{2}^\circ = \sqrt{2}+1.$$

Also $$\tan 22\tfrac{1}{2}^\circ = \sqrt{2}-1;\quad \tan 52\tfrac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}-1);$$

$$\tan 82\tfrac{1}{2}^\circ = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1);$$

therefore product $= (2\sqrt{6}+\sqrt{2}-3)(2\sqrt{6}+\sqrt{2}+3) = 17 + 8\sqrt{3}$.

[By easy reductions, the product can be brought to
$(5 + 6\cos 75^\circ)(5 - 6\cos 75^\circ)/(2\sqrt{2}\cos 75^\circ) =$ the given result.]

9195. (Sir JAMES COCKLE, F.R S.)—Integrate

$$\frac{d^3u}{dt^3} = \frac{u}{(a^2+b^2t^m)^3}, \text{ when } m = 1 \text{ or when } m = 2.$$

Solution by the PROPOSER.

1. Let $m = 1$; put $4t = b^2x^2 - c^2$; then $\dfrac{d^3u}{dx^3} + \frac{1}{8}\dfrac{u}{x^{\frac{3}{2}}} = 0$ is satisfied by

$$u = C_0(1+\sqrt{x})\,e^{-\sqrt{x}} + C_1\left(\frac{1}{\alpha}+\sqrt{x}\right)e^{-\alpha\sqrt{x}} + C_2\left(\frac{1}{\beta}+\sqrt{x}\right)e^{-\beta\sqrt{x}},$$

where α and β are the unreal cube roots of unity.

2. Let $m = 2$; put $t = n\sin\theta$ and $b^2n^2 = -a^2$; then

$$\left(\frac{1}{\cos\theta}\frac{d}{d\theta}\right)^3 u = \frac{u}{\mathrm{a}^3(\cos\theta)^3};\quad \frac{d}{d\theta}\left(\frac{1}{\cos\theta}\frac{d}{d\theta}\right)^2 u = \frac{u}{\mathrm{a}^3(\cos\theta)^2};$$

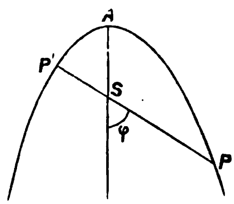
whence $\quad \dfrac{d^3u}{d\theta^3} + 3\tan\theta\dfrac{d^2u}{d\theta^2} + 3\left\{(\tan\theta)^2 + \frac{1}{3}\right\}\dfrac{du}{d\theta} = \dfrac{u}{\mathrm{a}^3}$; and, if $u' = \dfrac{du}{d\theta}$,

$$\frac{d^3u'}{d\theta^3} + 3\tan\theta\frac{d^2u'}{d\theta^2} + 3\left\{2(\tan\theta)^2 + \tfrac{4}{3}\right\}\frac{du'}{d\theta} + 3\left\{2\tan\theta\left[1+(\tan\theta)^2\right]\right\}u' = \frac{u'}{\mathrm{a}^3}.$$

Now put $u = (\cos\theta)\,y$; then $\dfrac{d^3y}{d\theta^3} + \dfrac{dy}{d\theta} - \dfrac{1}{a^3}y = 0$, whereof the coefficients are constant and wherein $\mathrm{a} = -b\sqrt{-1}$. The solution of case (1) is obtained by an analogous process.

8743. (C. BICKERDIKE.)—Prove that (1) the length of a focal chord of the parabola is $l\operatorname{cosec}^2\phi$; (2) when the chord is one of quickest descent, $\cos\phi = (\frac{1}{3})^{\frac{1}{2}}$; and (3) the time of quickest descent down the chord then is $\sqrt{(3^{\frac{3}{2}}l)/g}$, where l is the latus-rectum, and ϕ the angle made by the chord with the axis.

Solution by GEORGE GOLDTHORPE STORR, M.A.



1. $$\mathrm{SP}+\mathrm{SP}' = \frac{2a}{1-\cos\phi} + \frac{2a}{1+\cos\phi} = \frac{4a}{\sin^2\phi} = \frac{l}{\sin^2\phi}.$$

2. Here $t = \{2l/(\sin^2\phi\,.\,g\cos\phi)\}^{\frac{1}{2}}$ ... (α); and, as this is a minimum, $\sin^2\phi\cos\phi \equiv x - x^3$ (where $x \equiv \cos\phi$) must be a maximum, which is the case when $\cos\phi \equiv x = (\frac{1}{3})^{\frac{1}{2}}$.

Substituting this value in (α), we find the time of quickest descent to be as stated in the Question.

9381. (Professor SYLVESTER, F.R.S.) — If (q and r being prime numbers) $1+p+p^2+\dots p^{r-1}$ is divisible by q, show that, unless r divides $q-1$, it must be equal to q and divide $p-1$.

Solution by Professor GENESE, M.A.

Let N_t denote a number which is expressed in the scale of p by 111 ... to t digits. Now, q being a prime,

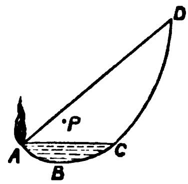
$$p^q-p = M(q), \quad i.e., \quad p(p-1)N_{q-1} = M(q);$$

and, by the question, $N_r = M(q)$; thus N_{q-1}, N_r have a common measure q, unless q divide $(p-1)$, for it clearly does not divide p.

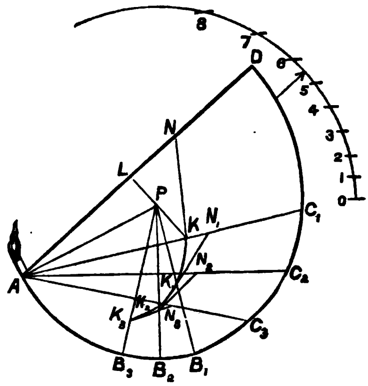
1. The arithmetical process shows at once that N_{q-1}, N_r cannot have a common factor unless of the form N_t, where t divides $q-1$ and r; but r is prime, therefore r divides $q-1$ (which is not prime).

2. If $p = mq+1$, we find $N_r = M(q)+(1+1+\dots$ to r terms), whence $q = r$.

2353. (The late Professor DE MORGAN.)—The late Dr. MILNER, President of Queens' College, Cambridge, constructed a lamp which General PERRONET THOMPSON remembered to have seen. It is a thin cylindrical bowl, revolving about an axis at P, and the curve ABCD is such that, whatever quantity of oil ABC may be in the bowl, the position of equilibrium is such that the oil just wets the wick at A. Find the curve ABCD.



Solution by D. BIDDLE.



Let AC_1, AC_2, AC_3 represent the surface of the oil at three different times; then in PB_1, PB_2, PB_3, perpendicular to these respectively, will lie the centre of gravity of the oil at those particular times. Consequently, the level of the oil, always flush with A, and the line joining the centre of gravity with P, describe equal angles in a given time.

Taking a wedge-shaped portion of oil, of infinitesimal depth, with its apex at A, and its base at C_1, its particular centre of gravity will be at $\frac{2}{3}AC_1$ from A, say at N_1; and, if K_1 be the centre of gravity of the mass of oil when at the level AC_1, then N_1K_1 will be tangential to the locus of the centre of gravity of the mass.

Let x_1, y_1 be the coordinates of the centroid K_1, x, y, the coordinates of the centroid K (that is, of the oil when the vessel is full).

Let $M =$ the mass having the centroid K_1, and x', y' be the coordinates of the centroid of AC_1D ($\equiv 1-M$) now empty; also let $\angle DAC_1 = a$.

Then $$PK_1 \sin a \, M = (x'-x)(1-M),$$

and $$(PK_1 \cos a - PK)\, M = (y-y')(1-M),$$

whence $PK_1 = \dfrac{y_1 - y + PK}{\cos a} = \dfrac{x-x_1}{\sin a}$, and $PK = \cot a\,(x-x_1)-(y_1-y)$.

If ABCD were a semi-circle, and P its centre, these conditions would be fulfilled, but as the oil sank in the vessel there would be no bias to cause rotation. For the purposes specified in the question, it is essential that, as the oil is consumed and its level sinks, the layer taken off shall have been unequally divided by the perpendicular from P, the lesser portion being always on the side next A, so that what remains is heavier on that side, until by rotation equilibrium is restored.

A very near approach to the curve required is given in the accompanying figure, where $r = \cos^{\frac12}\theta$, $dr/d\theta = \sin\theta/2r$, and the area of the side of the vessel $= \frac12 AD^2$. With this curve, if, as may be supposed, equal quantities of oil are consumed in equal times, an indicator parallel to AD and projecting from the vessel as if drawn from P, will rise at a uniform rate, because the sine of the angle DAC lengthens at a uniform rate. Such being the case, we have at once not only a convenient light, but also a time-keeper scarcely inferior to Alfred the Great's graduated candle.

The positions of K and P, though easy to find by practical methods, do not yield readily to the *integral calculus*.

But $$AL = \tfrac23 \int_0^{\frac12\pi} \cos^{\frac52}\theta\, d\theta + \int_0^{\frac12\pi} \cos\theta\, d\theta,$$

$$LK = \tfrac23 \int_0^{\frac12\pi} \cos^{\frac32}\theta \sin\theta\, d\theta + \int_0^{\frac12\pi} \cos\theta\, d\theta.$$

9386. (Professor NEUBERG.)—Si suivant les perpendiculaires abaissées du centre O du cercle circonscrit à un triangle ABC, sur les côtés de ce triangle, on applique, dans un sens ou dans l'autre, trois forces égales, la résultante passera par le centre de l'un des cercles tangents aux trois côtés.

Solution by Professors GENESE, M.A.; BEYENS; *and others.*

Let L, M, N be the mid-points of the sides, P, Q, R the points of contact of *any one* of the four circles touching the sides. Then we have

$$PL = \tfrac12 (PB + PC), \text{ \&c.};$$

and if, following LAGUERRE's principles, we define, for this case, the positive direction of the sides as that leaving the circle in the positive sense of rotation, we have

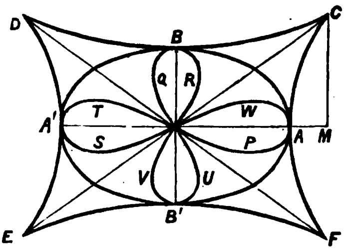
$$PC + QC = 0, \text{ \&c.}, \quad \therefore\ PL + QM + RN = 0.$$

Whence, for suitable directions of the forces in question, the sum of their moments about the centre of the circle is seen to be zero.

[Professor GENESE adds that this problem also occurred to, and was set by, him at Aberystwyth, in 1886; and that, if the equal forces be represented by radii of the circum-circle, the lines representing the resultants *terminate at* the centres in question.]

9315. (Professor MUKHOPÂDHYÂY, M.A., F.R.S.E.)—Prove that (1) the locus of the mid-points of the chords of curvature of the conic $b^2x^2 + a^2y^2 = a^2b^2$ is the sextic $\Sigma_1 \equiv a^{-2}x^2 + b^{-2}y^2 = (a^{-2}x^2 - b^{-2}y^2)^{\frac{2}{3}}$ passing through the origin; (2) the area of Σ_1 is half the area (A) of the ellipse; (3) the envelope of the chords of curvature of the same conic is the sextic $\Sigma_2 \equiv (a^{-2}x^2 + b^{-2}y^2 - 4)^3 + 27\,(a^{-}\ x^2 - b^{-2}y^2)^2 = 0$; (4) the area of $\Sigma_2 = \frac{3}{2}$A; (5) trace the locus Σ_1 and the envelope Σ_2, and show that they touch each other and the conic at the ends of the major and the minor axes.

Solution by Professor R. SWAMINATHA AIYAR, B.A.



1. The line $a^{-1}x\cos\phi - b^{-1}y\sin\phi = \cos 2\phi$...(i.) passes through the point (ϕ) on the ellipse, and makes the same angle with the axis that the tangent at the point does: it is therefore the chord of curvature at ϕ. The diameter conjugate to it is the line $a^{-1}x\sin\phi + b^{-1}y\cos\phi = 0$...(ii.); and, eliminating ϕ between (i.) and (ii.), we have Σ as the equation of the required locus.

2. From (i.) and (ii.), $x = \frac{1}{2}a(\cos\phi + \cos 3\phi)$, $y = \frac{1}{2}b(\sin\phi - \sin 3\phi)$,

$$\therefore \quad \int y\,dx = \tfrac{1}{4}ab\int(\sin 3\phi - \sin\phi)(3\sin 3\phi + \sin\phi)\,d\phi = \tfrac{1}{4}ab\int \mathrm{F}\,d\phi.$$

The required area $= ab\int_0^{\frac{1}{2}\pi}\mathrm{F}\,d\phi = \frac{1}{2}\pi ab = \frac{1}{2}$A.

3. Differentiating (i.), we have $a^{-1}x\sin\phi + b^{-1}y\cos\phi = 2\sin 2\phi$...(iii.). From (i.) and (iii.), $x = \frac{1}{2}a(3\cos\phi - \cos 3\phi)$, $y = \frac{1}{2}b(3\sin\phi + \sin 3\phi)$; and eliminating ϕ, we have Σ_2.

4. $\int y\,dx = \frac{3}{4}ab\int(\sin 3\phi + 3\sin\phi)(\sin 3\phi - \sin\phi)\,d\phi = \frac{3}{4}ab\int \mathrm{F}\,d\phi$; hence the required area of Σ_2 is $\frac{3}{2}ab\int_\pi^0 \mathrm{F}\,d\phi = \frac{3}{2}\pi ab = \frac{3}{2}$A.

5. As ϕ increases from 0 to 2π, the locus Σ_1 is traced in the order PQRSTUVW; and the envelope Σ_2 in the order ACBDA'EB'FA. The points ($\pm a\sqrt{2}$, $\pm b\sqrt{2}$) are cusps in the latter curve, the equi-conjugate diameters of the ellipse being the cusp-tangents. In the first curve the origin is what might be called a double tacnode.

8989. (Professor WOLSTENHOLME, M.A., Sc.D.)—In a tetrahedron OABC, $OA = a$, $OB = b$, $OC = c$; $BC = x$, $CA = y$, $AB = z$, and the dihedral angles opposite to these edges are respectively A, B, C; X, Y, Z. Having given the equations $b = y = \frac{1}{2}(a+x)$, $c-z = a-x$, $B = Y$, $C+Z = 180°$, prove that $B = Y = 60°$, $C-A = Z-X = 30°$; and find the relations between a, b, c.

Solution by SEPTIMUS TEBAY, B.A.

We have $x = 2b-a$, $y = b$, $z = c+2b-2a$; and therefore

$BC+CO = AB+AO$, and $BC-BO = AC-AO$; or (Quest. 8605)

$A+Z = X+C$, and $A-Y = B-X$; and since $B = Y$, and $C+Z = 180°$,

therefore $$A = B+C-90°, \quad X = B-C+90°,$$

and $$\sin A \sin X = 1-\cos^2 B-\cos^2 C.$$

Let the areas of the faces BOC, COA, AOB, ABC be denoted by A_1, A_2, A_3, A_4; then

$$\frac{\sin B}{\sin A} = \frac{\sin BCO}{\sin ACO} = \frac{b}{x} \cdot \frac{A_1}{A_2}, \quad \frac{\sin B}{\sin X} = \frac{\sin AOC}{\sin BOC} = \frac{b}{a} \cdot \frac{A_2}{A_1};$$

$$\therefore \sin A \sin X = \frac{ax}{b^2} \cdot \sin^2 B = 1-\cos^2 B-\cos^2 C, \quad \therefore \cos C = \frac{b-a}{b} \sin B.$$

Also $$\frac{\sin B}{\sin C} = \frac{\sin ABC}{\sin CBO} = \frac{b}{z} \cdot \frac{A_4}{A_1} = \frac{\sin BCO}{\sin ACB} = \frac{A_1}{A_4}.$$

Therefore $\sin^2 C = \frac{cz}{b^2} \cdot \sin^2 B$. These equations give

$$\sin B = \frac{b}{b+c-a} = \frac{b}{c+\alpha}, \quad \sin C = \frac{b-a}{b+c-a} = \frac{\alpha}{c+\alpha} \quad \ldots\ldots\ldots(1, 2),$$

putting $b-a = \alpha$.

Now the general relation among the dihedral angles of a tetrahedron is

$\Sigma(\sin^2 A \sin^2 X) - 2\Sigma(\cos X \cos Y \cos Z) - 2\Sigma(\cos B \cos C \cos Y \cos Z) = 2$;

which in the present case reduces to

$$8\cos^2 B\cos^2 C - 4\cos^2 B - 4\cos^2 C + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

From (1, 2, 3), we have

$$4\sin^2 B = 2-\sec 2C = 3+\frac{2\alpha^2}{(c+\alpha)^2-2\alpha^2} = \frac{4b^2}{(c+\alpha)^2}.$$

This equation reduces to

$$\left.\begin{aligned} a^4+4(c-b)a^3-2(7c^2+6bc-b^2)a^2+4(3c^3+7bc^2+b^2c+b^3)a \\ = 3b^4-4b^3c+10b^2c^2+12bc^3+3c^4 \end{aligned}\right\} \ldots\ldots(4),$$

which is the general relation among a, b, c.

Thus the least value of B is 60°, which makes

$$C = Z = 90°, \quad a = b = x = y, \text{ and } 2a = c\sqrt{3}.$$

In the other cases we must have

$$\sin B = \frac{b}{b+c-a} > \cdot 8660254 < 1.$$

We have assumed $c > b > a$. Now $b/(b+c-a)$ is less than $c/(2c-a)$; and since c is the greatest value of b, B will be a maximum when $\sin B = c/(2c-a)$. Let $b=c=1$; then, from (4), $a^4-24a^2+48a-24=0$; from which we find $a = \cdot8518704$. These values of a, b, c make

$$B = Y = 60°34'22''\cdot76, \text{ and } C = 82°35'13''\cdot11.$$

If a be small in comparison with c, so that a^3 and higher powers may be neglected, we find

$$2b = c\sqrt{3}\left(1+\frac{a}{c}+\frac{a^2}{3c^2}\right), \text{ and } \sin B = \tfrac{1}{2}\sqrt{3}\left(1+\frac{a^2}{3c^2}\right),$$

which is not sensibly affected if $a =$ or $< \cdot001$. Hence any solution depending upon a small value of a makes $A+X = 120°$, nearly; and since $C+Z = 180°$, therefore $C-A+Z-X = 60°$, or $C-A=Z-X=30°$, since $C-A = Z-X$.

For the maximum value of B we have

$a = \cdot8518704$	$x = 1\cdot1481296$	$B = 60°34'22''\cdot76$
$b = 1$	$y = 1$	$C = 82°35'13''\cdot11$
$c = 1$	$z = 1\cdot2962592$	

$$A = 53°9'35''\cdot87, \quad X = 67°59'9''\cdot65.$$

[There being 5 equations given, apparently independent, it would seem that the shape of the tetrahedron must be fixed, but there is certainly more than one solution. One obvious solution is when $a = x = b = y$, when it will be found that $c\sqrt{3} = 2a = Z$, $C=Z=90°$, $A=X=B=Y=60°$. The tetrahedron in which

$$\begin{cases} a = 4\cdot8023, & x = 4\cdot8044 \\ b = 4\cdot80335, & y = 4\cdot80335 \\ c = 5\cdot64538, & z = 5\cdot54738 \end{cases}$$

satisfies the conditions; and
$A = 59°\ 59'\ 20''\cdot96 = C-60°$,
$X = 60°\ \ 0'\ 39''\cdot04 = Z-60°$,
$B = Y = 60°$].

8701. (A. Russell, B.A.)—Resolve into quadratic factors

$$(a^2-bc)^5(b+c)^5(b-c)\left\{a^2+2a(b+c)+bc\right\}$$
$$+(b^2-ca)^5(c+a)^5(c-a)\left\{b^2+2b(c+a)+ca\right\}$$
$$+(c^2-ab)^5(a+b)^5(a-b)\left\{c^2+2c(a+b)+ab\right\}.$$

Solution by R. F. Davis, M.A.; Professor Beyens; *and others.*

Let $A = (a^2-bc)(b+c)$, $B = \ldots$, $C = \ldots$; so that $A+B+C = 0$.
Then, since $\quad B-C = (b-c)\left\{a^2+2a(b+c)+bc\right\}$,
the given expression may be written $A^5(B-C)+\ldots+\ldots$, which can easily be reduced to the form $-ABC(B-C)(C-A)(A-B)$. The given expression (which is homogeneous and of the 18th degree) is therefore equal to the product (with its sign changed) of nine quadratic factors; three of the form b^2-c^2, three of the form a^2-bc, and three of the form $a^2+2a(b+c)+bc$.

7759. (Professor HANUMANTA RAU, M.A.) — From one end A of the diameter AB ($=2a$) of a semicircle, a straight line APMN is drawn meeting the circumference at N, and a given straight line through B at M, at an angle α; show that the locus of a point P, such that AP, AM, AN are proportionals, is the cubic through A,

$$r = 2a\sin^2\alpha \sec\theta \operatorname{cosec}^2(\alpha-\theta), \text{ or } 2a\sin^2\alpha\,(x^2+y^2) = (x\sin\alpha - y\cos\alpha)^2,$$

which, when $\alpha = \frac{1}{2}\pi, \frac{1}{4}\pi$, becomes

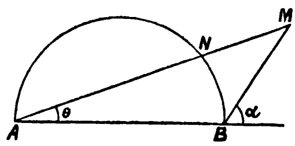
$$2a^2(x^2+y^2) = x^3, \quad 2a^2(x^2+y^2) = x(x-y)^2.$$

Solution by G. G. STORR, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

The polar equations of the line and the circle are respectively

$$r = 2a\sin\alpha \operatorname{cosec}(\alpha-\theta), \quad r = 2a\cos\theta.$$

But $AP . AN = AM^2$; hence the locus of P is given by the equations stated in the Question.



8852. (J. GRIFFITHS, M.A.)—If $\alpha, \beta, \gamma, \delta$ be the roots of the quartic $ax^4+4bx^3+6cx^2+4dx+e=0$, and if $q = \dfrac{\alpha-\gamma}{\alpha-\delta} + \dfrac{\beta-\gamma}{\beta-\delta}$; show that

$$\frac{(q^2-q+1)^3}{(2-q)^2(1-2q)^2(1+q)^2} = \frac{I^3}{108J^2},$$

where $\quad I = ae-4bd+3c^2, \quad J = ad^2+eb^2+c^3-ace-2bcd.$

Solution by D. EDWARDES; G. G. STORR, M.A.; *and others.*

Let the quartic be linearly transformed into $a'(1-mx^2)(1-nx^2)$, and, as in the *Fundamenta Nova*, let

$U-\gamma V = A(1+m^{\frac{1}{2}}x)$, $U-\delta V = B(1-m^{\frac{1}{2}}x)$, $U-\alpha V = C(1+n^{\frac{1}{2}}x)$, $U-\beta V = D(1-n^{\frac{1}{2}}x)$. Putting $x = -n^{-\frac{1}{2}}, +n^{-\frac{1}{2}}$ successively in these, we get $\quad q = \dfrac{\alpha-\gamma}{\alpha-\delta}+\dfrac{\beta-\gamma}{\beta-\delta} = \left(\dfrac{n^{\frac{1}{2}}-m^{\frac{1}{2}}}{n^{\frac{1}{2}}+m^{\frac{1}{2}}}\right)^2$, giving $\dfrac{m}{n} = \left(\dfrac{1-q^{\frac{1}{2}}}{1+q^{\frac{1}{2}}}\right)^2$.

But we have $\quad \dfrac{(m^2+n^2+14mn)^3}{(m+n)^2(34mn-m^2-n^2)^2} = \dfrac{I^3}{27J^2}$

(CAYLEY'S *Elliptic Functions*, Arts. 413—14); hence substituting for m/n in terms of q, we have the stated result.

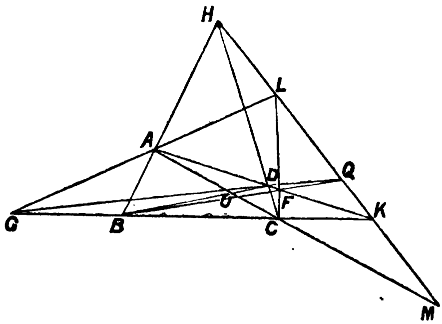
8850. (W. J. GREENSTREET, B.A.)—Prove that the sum of all the harmonic means which can be inserted between all the pairs of numbers whose sum is n, is $\frac{1}{6}(n^2-1)$.

Solution by A. W. CAVE, M.A.; W. J. BARTON, M.A.; *and others.*

$$\begin{aligned} \text{Sum} &= 2\left[\frac{n-1}{n}+\frac{2(n-2)}{n}+\frac{3(n-3)}{n}+\ldots\right] \\ &= 2\left[1+2+\ldots+(n-1)\right]-2/n\left[1^2+2^2+\ldots+(n-1)^2\right] \\ &= n(n-1)-\tfrac{1}{3}(n-1)(2n-1) = \tfrac{1}{3}(n^2-1). \end{aligned}$$

9340. (R. KNOWLES, B.A.)—In Question 9149, if BD and AC intersect in O, and CA meet KH in M; prove that the lines GM, GA, GO, GB and LC, LO, LA, LH form harmonic pencils.

Solution by G. G. MORRICE, M.A., M.B.

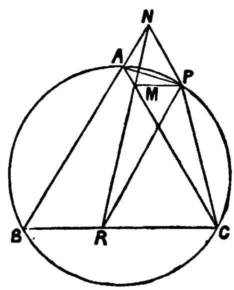


$$G\{M.A.O.B\}$$
$$= G\{M.A.O.C\}$$
$$= H\{M.A.O.C\},$$

which is an harmonic pencil, by the known property of a complete quadrilateral; similarly

$$L\{C.O.A.H\}$$
$$= H\{C.O.A.M\}.$$

8300. (Professor HANUMANTA RAU, M.A.)—From any point P on the circle described about an equilateral triangle ABC, straight lines PM, PN, PR are drawn respectively parallel to BC, CA, and AB, and meeting the sides CA, AB, BC at M, N, and R. Prove that the points M, N, R are collinear.



Solution by D. O. S. DAVIES, B.A.; R. KNOWLES, B.A.; *and others.*

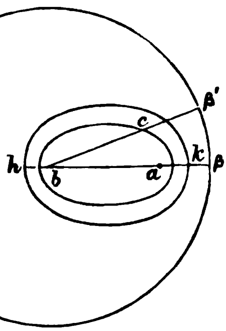
Since ABC is an equilateral triangle, evidently ANP, PMC, PRC are angles of equilateral triangle. Hence N, A, M, P and P, M, R, C are concyclic. Join PA, PC, NM, and MR; then

$$\angle PMN = PAN = PCB;$$

hence N, M, R are collinear.

4251. (Colonel CLARKE, C.B., F.R.S.)—If A, B, C be three circles, B being within A, and C within B; prove that the chance that the centre of A is within C is $\frac{1}{7}$.

Solution by the PROPOSER.



In the accompanying diagram, let a, b, c be the centres of A and of B, C, two circles fulfilling the condition of which the probability is required. It will be convenient to denote such circles by accented letters B′C′. Let $(C')_{xy}$ represent the number of circles C′ within B′ of radius $b\beta = y$ and for which $ab = x$; while (C') the entire number when B′ has taken all magnitudes and positions. Then, lastly, if (C) denote the entire number of C's, the chance required is $(C')/(C)$.

It is easy to see that the centre c of any C′ cannot fall outside the ellipse whose foci are a, b and major axis y = radius of B′, for this ellipse is the locus of the centre of those circles which, passing through a, touch B′. Let c be on an interior confocal ellipse whose minor and major axes are z and $v = (x^2+z^2)^{\frac{1}{2}}$; then the number of the circles C′ with centre c is $c\beta' - ca = y - v$. And so, if c be any point in the elementary area $\frac{1}{4}\pi d\,(vz)$ contained between two consecutive confocal ellipses, the number of circles C′ whose centres are in that area is $\frac{1}{4}\pi\,(y-v)\,d\,(vz)$. This integrated from $z = 0$ to $z = (y^2-x^2)^{\frac{1}{2}}$ gives us $(C')_{xy}$. Now we have

$$\int (y-v)\,d\,(vz) = (y-v)\,vz + \int vz\,dv,$$

which between the limits, and since $v\,dv = z\,dz$, becomes simply $\frac{1}{3}z^3$; that is,

$$(C')_{xy} = \tfrac{1}{12}\pi\,(y^2-x^2)^{\frac{3}{2}}.$$

We may include all the circles B′, corresponding to x and y, by multiplying this by the area contained between the circle whose centre is a with radius x and the consecutive concentric circle whose area $= d\,(\pi x^2)$. Hence, between the proper limits, $(C') = \frac{1}{6}\pi^2 \iint (y^2-x^2)^{\frac{3}{2}}\,x\,dx\,dy.$

Taking unity as the radius of A, $x < \frac{1}{2}$, and $y > x$ and $< 1-x$, that is,

$$(C') = \tfrac{1}{6}\pi^2 \int_0^{\frac{1}{2}} \int_x^{1-x} (y^2-x^2)^{\frac{3}{2}}\,x\,dx\,dy \;\ldots\ldots\ldots\ldots\ldots\ldots(a),$$

the y integration is to be first effected. After some reduction and a substitution $1-2x = u^2$, we get, after the second integration, $(C') = \pi^2/7\cdot180$. It is, of course, immaterial which integration is performed first. If we commence with the x integration, the integral (a) is transformed into

$$\tfrac{1}{6}\pi^2 \int_0^{\frac{1}{2}} \int_0^{y} (y^2-x^2)^{\frac{3}{2}}\,x\,dy\,dx + \tfrac{1}{6}\pi^2 \int_{\frac{1}{2}}^{1} \int_0^{1-y} (y^2-x^2)^{\frac{3}{2}}\,x\,dy\,dx.$$

The first double integral embraces all circles B′ for which the radius is less than $\frac{1}{2}$; the second, those having a radius greater than $\frac{1}{2}$; the separate values are easily found to be

$$\frac{\pi^2}{180}\cdot\frac{1}{2^6} \text{ and } \frac{\pi^2}{180}\left(\frac{1}{7}-\frac{1}{2^6}\right).$$

But (a) is most easily integrated by taking two new variables such that $\xi\sqrt{2}=x+y$, $\eta\sqrt{2}=y-x$. Thus we have, instead of (a),

$$(C') = \tfrac{1}{3}\pi^2\int_0^{1/\sqrt{2}}\int_0^{\xi}(\xi^{\frac{5}{2}}\eta^{\frac{3}{2}}-\xi^{\frac{3}{2}}\eta^{\frac{5}{2}})\,x\xi\,d\eta = \tfrac{1}{3}\pi^2\int_0^{1/\sqrt{2}}(\tfrac{2}{5}-\tfrac{2}{7})\,\xi^5\,d\xi,$$

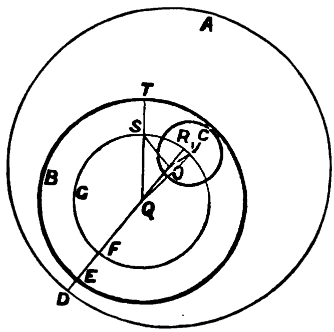
giving immediately the result as before, viz., $\pi^2/7\cdot180$.

Finally the number of C's within a B of radius y is

$$2\pi\int r\,dr\,(y-r) = \tfrac{1}{3}\pi y^3; \text{ thus we have}$$

$$(C) = \tfrac{2}{3}\pi^2\int_0^1\int_0^{1-x} xy^3\,dx\,dy = \tfrac{1}{6}\pi^2\int_0^1 x\,(1-x)^4\,dx = \tfrac{1}{6}\pi^2\int_0^1(x^4-x^5)\,dx = \frac{\pi^2}{180}.$$

This gives the required chance $=\dfrac{1}{7}$.



[Assuming that in each case the position of the centre is first taken, and then the radius, within the proper limits, Mr. Biddle proceeds to find the required probability as follows:—

"Let O, Q, V be the centres of A, B, C respectively; let OD = 1,

$$OQ = x,\quad QT\,(=QE) = y,$$
$$QS\,(=QF) = z.$$

Then we readily discover the following limits for success:—

$$\text{zero} < x < \tfrac{1}{2},\quad x < y < (1-x),$$
$$\text{zero} < z < \tfrac{1}{2}\,(x+y).$$

Make $OS = ST = EF = y-z$, and join QV (= QS); also join OV, and let $\angle OQV = \phi$. Then we have the further limits, $\text{zero} < \phi < SQO \left(=\cos^{-1}\dfrac{x^2+2yz-y^2}{2xz}\right)$,

and $OV < u$ ($\equiv$ rad. of C) $< (y-z)$, which may be rendered

$$(x^2+z^2-2xz\cos\phi)^{\frac{1}{2}} < u < (y-z)."$$

By this process he arrives at an integral very troublesome to evaluate.]

7986. (J. Brill, B.A.)—ABCD is a quadrilateral, AB and DC when produced meet in E, and AD and BC when produced meet in F; prove that

$$\begin{aligned}&AB\,.\,CE\,.\,DF\cos(ABD+CEF+CAF)\\ &+AD\,.\,CF\,.\,BE\cos(ADB+CFE+CAE)\\ &-BC\,.\,AF\,.\,DE\cos(CFE+ADB+DCA)\\ &-CD\,.\,AE\,.\,BF\cos(CEF+ABD+BCA) = AC\,.\,BD\,.\,EF.\end{aligned}$$

Solution by A. GORDON; Professor NASH, M.A.; *and others.*

Let i, j, k be unit vectors along Ox, Oy, Oz, and $\alpha, \beta, \gamma, \delta$, &c. unit vectors in the plane of xy, such that

$\alpha = i\cos\alpha + j\sin\alpha$ (inclined at $\hat{a}$ to Ox, &c.), $\beta = i\cos\beta + j\sin\beta$, &c.;

then $\qquad \alpha/\beta = \cos(\alpha-\beta) + k\sin(\alpha-\beta)$, &c.,

$$\alpha/\beta\,.\,\gamma/\delta = \cos(\alpha+\gamma-\beta-\delta) + k\sin(\alpha+\gamma-\beta-\delta) = \alpha/\delta\,.\,\gamma/\beta;$$

hence, if

$\alpha_1 = l_1\beta + l_2\gamma + l_3\delta + \ldots,\quad \alpha_2 = m_1\beta + m_2\gamma + m_3\delta + \ldots,\quad \alpha_3 = n_1\beta + n_2\gamma + n_3\delta + \ldots,$

and $\theta, \phi, \psi, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \delta \ldots$ are all vectors in the plane of xy, we have $\quad \alpha_1/\theta\,.\,\alpha_2/\phi\,.\,\alpha_3/\psi = \alpha_1/\phi\,.\,\alpha_2/\theta\,.\,\alpha_3/\psi = \&\text{c.} = \Sigma_{lmn}^{lmn}\,\beta/\theta\,.\,\gamma/\phi\,.\,\delta/\psi,$

where Σ constitutes a summation of terms each formed of the product of 3 quaternions, such as $l_1\beta/\theta\,.\,m_2\gamma/\phi\,.\,n_3\delta/\psi$ (one from each vector), (the three numerators being the same in none of the products).

Let large letters denote *vectors*, small letters *lengths*, so that

$$AB = -BA, \text{ but } ab = ba; \text{ then}$$

$$1 = \frac{AF-AE}{EF} = \frac{AF}{EF}\cdot\frac{AC}{AC} - \frac{AE}{EF}\cdot\frac{AC}{AC}$$

$$= \frac{AF\,.\,AC + AF\,.\,AE - AE\,.\,AF - AE\,.\,AC}{EF\,.\,AC}$$

$$= \frac{AE\,.\,CF - AF\,.\,CE}{EF\,.\,AC} = \frac{AE}{AC}\left(\frac{BF-BC}{EF}\right) - \frac{AF}{AC}\left(\frac{CD+DE}{EF}\right)$$

$$= \frac{AE\,.\,BF\,.\,CD + CE\,.\,DF\,.\,BA + FC\,.\,BE\,.\,AD + FA\,.\,DE\,.\,BC}{AC\,.\,EF\,.\,BD};\text{ hence}$$

$$1 = \frac{BA}{BD}\cdot\frac{CE}{EF}\cdot\frac{DF}{AC} + \frac{AD}{BD}\cdot\frac{BE}{AC}\cdot\frac{FC}{EF} + \frac{DE}{AC}\cdot\frac{FA}{BD}\cdot\frac{BC}{EF} + \frac{AE}{BD}\cdot\frac{CD}{EF}\cdot\frac{BF}{AC};$$

Also, by Ptolemy's theorem, $\quad CD\,.\,BE\,.\,AF = CE\,.\,BA\,.\,DF,$

and $\qquad BC\,.\,AE\,.\,DF = FC\,.\,BE\,.\,AD;$

therefore $\qquad 1 = \dfrac{ba}{bd}\cdot\dfrac{ec}{ef}\cdot\dfrac{df}{ac}\left\{\cos(ABD+CEF+FAC) + k\sin(ABD+CEF+FAC)\right\} + \ldots.$

Hence $\qquad bd\,.\,ef\,.\,ac = \Sigma\, ba\,.\,ec\,.\,df\,.\cos(ABD+CEF+FAC),$

the result required,

and $\qquad O = \Sigma\, ba\,.\,ec\,.\,df\sin(ABD+CEF+FAC).$

8771. (W. J. GREENSTREET, B.A.)—Prove that the series

$$U \equiv \sin\alpha\left\{\tfrac{1}{2} + \frac{1}{2\,.\,4}\sin^2\alpha + \frac{1\,.\,3}{2\,.\,4\,.\,6}\sin^4\alpha + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}\sin^6\alpha + \ldots\right\} = \tan\tfrac{1}{2}\alpha.$$

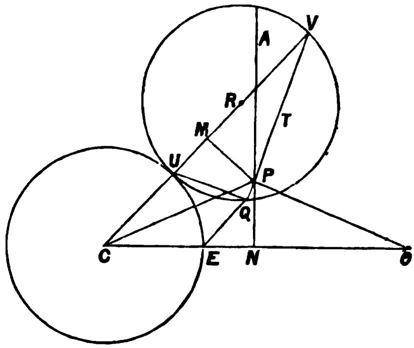
Solution by Prof. IGNACIO BEYENS; R. KNOWLES, B.A; *and others.*

$$\frac{du}{d\alpha} = \cos\alpha\left\{\tfrac{1}{2} + \frac{1\,.\,3}{2\,.\,4}\sin^2\alpha + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\sin^4\alpha + \ldots\right\}$$

$$= \frac{\cos\alpha}{\sin^2\alpha}\left\{(1-\sin^2\alpha)^{-\frac{1}{2}} - 1\right\} = \tfrac{1}{2}\sec^2\tfrac{1}{2}\alpha;\ \text{therefore } u = \tan\tfrac{1}{2}\alpha.$$

9327. (F. R. J. Hervey.) — The point O is fixed, P describes a straight line A; OP and a line T passing through P rotate uniformly (in the same or in contrary senses) with angular velocities as 1 : 3, and become simultaneously perpendicular (or, in the limiting position, parallel) to A. Show that the envelope of T is a cardioid.

Solution by R. F. Davis, M.A.; G. G. Storr, M.A.; *and others.*



Let the circle (centre R) roll upon an equal circle (centre C), U being the point of contact and CURV the common diameter through U. Then, if Q be an invariable point on the moving circle, QV and QU are the tangent and normal respectively to the cardioid described by Q, whose cusp is at E, the point passed over by Q when in contact with the fixed circle. Since arc UE = arc UQ, angle

UCE = URQ = 2UVQ;

so that, if CP bisect the angle UCE, CP = PV. Draw PN, PM perpendicular to CE, CR respectively, and take CO = CV = constant. Then the angles OPN, CPN, OPM, VPM are all equal, and VP may be conceived as having revolved from PN through an angle three times as great as that revolved through by OP from PN in the opposite direction.

8737. (Professor Mukopâdhyây, M.A., F.R.S.E. — Extension of Question 8107.)—If θ, ϕ, ψ be the angles of inclination of any two tangents to a conic, and of their chord of contact, to a directrix, show that, if e be the eccentricity of the conic,

$$\tan\psi = \frac{\lambda^{-1}\sin\theta + \mu^{-1}\sin\phi}{\lambda^{-1}\cos\theta + \mu^{-1}\cos\phi}, \qquad e^2 = \frac{1-\lambda^2}{\sin^2\theta} = \frac{1-\mu^2}{\sin^2\phi}.$$

Solution by R. Knowles, B.A.; Professor Matz, M.A.; *and others.*

Let (x_1, y_1), (h, k) be the coordinates of the ends of the chord and its pole, then
$$\tan\psi = \frac{a^2 k}{b^2 h}, \quad \cot\theta = +\frac{b^2 x_1}{a^2 y_1}, \quad \cot\phi = \frac{b^2 x_2}{a^2 y_2},$$

therefore
$$\sin^2\theta = \frac{a^2 y_1^2}{b^2 (a^2 - e^2 x_1^2)^{\frac{1}{2}}}; \quad \text{and, since } e^2 = \frac{1-\lambda^2}{\sin^2\theta},$$

therefore $$\lambda = \frac{b}{(a^2-e^2x_1^2)^{\frac12}} \text{ and } \mu = \frac{b}{(a^2-e^2x_2^2)^{\frac12}},$$

and $$\frac{\lambda^{-1}\sin\theta+\mu^{-1}\sin\phi}{\lambda^{-1}\cos\theta+\mu^{-1}\cos\phi} = \frac{a^2(y_1+y_2)}{b^2(x_1+x_2)} = \frac{a^2k}{b^2h} = \tan\psi.$$

9353. (Professor ÂSUTOSH MUKHOPÂDHYÂY, M.A., F.R.S E.)—Points D, E are taken in the sides AB, BC of any triangle ABC, such that BD $= m$. DA, BE $= n$. EC. If O be the intersection of AE, DC, prove

that $$\frac{\text{CO}}{\text{OD}} = \frac{m+1}{n} \text{ and } \frac{\text{AO}}{\text{OE}} = \frac{n+1}{m}.$$

Solution by R. F. DAVIS, M.A.; Professor W. P. CASEY, M.A.; *and others.*

The point O is (by hypothesis) the centroid of masses m, 1, n placed at A, B, C respectively. Whence, &c.

9089. (EMILE VIGARIÉ.)—Par les sommets A, B, C d'un triangle on mène des parallèles aux côtés opposés qui rencontrent le cercle circonscrit en A′, B′, C′. Les droites A′B′, A′C′, C′B′ rencontrent respectivement AB, AC, BC en α, β, γ. Démontrer que l'orthocentre du triangle $\alpha\beta\gamma$ est le centre du cercle ABC.

Solution by R. KNOWLES, B.A.; Professors MATZ, M.A.; *and others.*

The equations to AA′, BB′, CC′ and the coordinates of A′B′C′ are respectively $by+cz=0$, $ax+cz=0$, $ax+by=0$; $1/a$, $(c^2-b^2)/a^2b$, $(b^2-c^2)/a^2c$; $(c^2-a^2)/ab^2$, $1/b$, $(a^2-c^2)/(b^2c)$; $(b^2-a^2)/ac^2$, $(a^2-b^2)/b^2c$, $1/c$ (omitting 2Δ in all the coordinates); hence we find the equations to A′B′, A′C′, B′C′, and also the coordinates of $\alpha\beta\gamma$:—

$$(a^2-c^2)\,/\,a\,(a^2-b^2),\quad (c^2-b^2)\,/\,b\,(a^2-b^2),\quad 0;$$
$$(b^2-a^2)\,/\,a\,(c^2-a^2),\ 0,\ (c^2-b^2)\,/\,c\,(c^2-a^2);\ 0,\ (b^2-a^2)\,/\,b\,(b^2-c^2),$$
$$(a^2-c^2)\,/\,c\,(b^2-c^2);$$

therefore the equations to $\alpha\beta$ and the perpendiculars from γ, β on $\alpha\beta$, $\alpha\gamma$ are

$$a\,(c^2-b^2)\,x+b\,(c^2-a^2)\,y+c\,(a^2-b^2)\,z=0,$$

$$a\left[(b^2+c^2)\cos A-bc\right]x+b\,(a^2-c^2)\cos Ay+c\,(a^2-b^2)\cos Az=0\ \ldots\ldots(1),$$

$$a\,(b^2-c^2)\cos Bx+b\left[(a^2+c^2)\cos B-ac\right]y+c\,(b^2-a^2)\cos Bz=0\ \ldots\ldots(2),$$

and (1), (2) are each satisfied by $R\cos A$, $R\cos B$, $R\cos C$; hence the orthocentre of the triangle $\alpha\beta\gamma$ is the centre of the circle ABC.

8667. (N'Importe.)—Two equal perfectly elastic balls, moving in directions at right angles to each other, impinge, their common normal at the instant of impact being inclined at any angle to the directions of motion : show that, after impact, the directions of motion will still be at right angles.

Solution by F. R. J. Hervey; Rev. T. Galliers, M.A.; *and others.*

Let OA, OB represent the velocities, either before or after impact, and C be mid-point of AB. Then OC, velocity of mass centre, is invariable; and so (elasticity being perfect) is *magnitude* of relative velocity, or *length* AB. But, if AOB be a right angle, length AB = 2 × length OC; and the converse.

9360. (R. Curtis, M.A.)—A tetrahedron ABCD is circumscribed to an ellipsoid, and straight lines are drawn through the centre from the corners to the opposite sides meeting them in X, Y, Z, W; show that

$$\frac{OX}{XA}+\frac{OY}{YB}+\frac{OZ}{ZC}+\frac{OW}{WD}=1.$$

Solution by J. O'Byrne Croke, M.A.

Let P_1, p_1, P_2, p_2, P_3, p_3, P_4, p_4, s_1, s_2, s_3, s_4 be the parallel perpendiculars from the angles of the tetrahedron, and the point O, and the areas of the sides on which they respectively fall; then

$p_1s_1+p_2s_2+p_3s_3+p_4s_4 = 3$ times vol. of tetrahedron $= P_1s_1 = P_2s_2 = P_3s_3 = P_4s_4$.

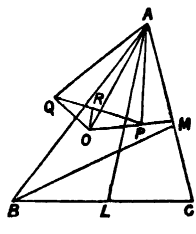
Therefore $\qquad \dfrac{p_1}{P_1}+\dfrac{p_2}{P_2}+\dfrac{p_3}{P_3}+\dfrac{p_4}{P_4}=1$, whence the result.

8270. (D. Edwardes.)—Let ABC be an acute-angled triangle, and L, M, N the points where the angle bisectors meet BC, CA, and AB respectively. Prove that (1) the circles ALB, ALC cut one another at an angle A, the circles ALC, ANC at an angle $\pm\frac{1}{2}(C-A)$, and the circles ALC, BNC at an angle $90^\circ-\frac{1}{2}B$; (2) the centres of the pair of circles which pass through L are equidistant from the centre of the circle ABC, and similarly for the other two pairs; (3) if ρ_L, ρ'_L be the radii and δ_L the distance between the centres of the circles which pass through L, and similarly for ρ_M, ρ'_M, &c., $\rho_L\rho_M\rho_N=\rho'_L\rho'_M\rho'_N=\delta_L\delta_M\delta_N$; (4) if d_1 be the distance of the circle ALB (or ALC) from the centre of the circle ABC (radius R), and similarly for d_2, d_3, $R^3-R(d_1d_2+d_2d_3+d_3d_1)-2d_1d_2d_3=0$; (5) if the base BC and the circum-circle BAC be given, the envelope of the line joining the centres of the circles ALB, ALC is a parabola whose focus is at the centre of the given circle and latus rectum $4R\sin^2\frac{1}{2}A$.

Solution by Professor SWAMINATHA AIYAR, B.A.

(1.) Let O, P, Q be the circum-centres of ABC, ALC, ALB; then OQ, OP are at right angles to AB, AC respectively, therefore QOP and BAC are supplementary, also

$$\angle AQO = ALC; \quad \text{and} \quad \angle APO = ALB.$$



Therefore QOP and QAP are supplementary and a circle passes through Q, A, P, O. Also $\angle QAP = BAC$. Again, $AP : AQ = AC : AB$, and ΔQAP is similar to ΔBAC, and the straight line AO bisects the angle QAP.

(2) Therefore QO, OP, the chords of the circle QAPO, are equal; that is, P and Q are equidistant from O.

(3) $\dfrac{AQ}{AB} = \dfrac{AP}{AC} = \dfrac{QP}{BC}$; that is, $\dfrac{\rho'_L}{c} = \dfrac{\rho_L}{b} = \dfrac{\delta_L}{a}$. Similarly,

$$\frac{\rho'_M}{b} = \frac{\rho_M}{a} = \frac{\delta_M}{c} \quad \text{and} \quad \frac{\rho'_N}{a} = \frac{\rho_N}{c} = \frac{\delta_N}{b}.$$

Therefore $$\rho_L \rho_M \rho_N = \rho'_L \rho'_M \rho'_N = \delta_L \delta_M \delta_N.$$

(4) $AO.QP = AQ.PO + AP.QO = (AQ+AP)\,d_1$. Therefore

$$\frac{R}{d_1} = \frac{b+c}{a}. \quad \text{Similarly } \frac{R}{d_2} = \frac{c+a}{b} \text{ and } \frac{R}{d_3} = \frac{a+b}{c}.$$

Therefore $$\frac{R^3}{d_1 d_2 d_3} = \frac{b+c}{a}\cdot\frac{c+a}{b}\cdot\frac{a+b}{c}$$

$$= \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} + 2 = \frac{R}{d_1} + \frac{R}{d_2} + \frac{R}{d_3} + 2.$$

(5) If R be the middle point of PQ, OR is at right angles to PQ.

The distance of R from BC $= \frac{1}{2}$ the distance of P and Q

$$= \tfrac{1}{2}\left(\frac{LC \cot\frac{1}{2}A + LB \cot\frac{1}{2}A}{2}\right) = \tfrac{1}{4}a \cot\tfrac{1}{2}A.$$

The locus of R is thus a straight line parallel to BC. Therefore PQ touches a parabola whose focus is O and whose tangent at the vertex is the locus of R.

Its latus rectum = 4, distance of O from the locus of R

$$= 4\left(\tfrac{1}{4}a \cot\tfrac{1}{2}A - \tfrac{1}{2}a \cot A\right) = a \tan\tfrac{1}{2}A = 4R\sin^2\tfrac{1}{2}A.$$

8540. (Rev. T. R. TERRY, M.A.)—Show that the series

$$1 + m\frac{q}{p} + \frac{m(m+1)}{1.2}\,\frac{q(q+r)}{p(p+r)} + \frac{m(m+1)(m+2)}{1.2.3}\,\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)} + \dots$$

is convergent if $p > q + mr$.

Solution by the PROPOSER; Professor NASH, M.A.; *and others.*

Denoting the hyper-geometric series

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}\,x+\frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{1\,.\,2\,.\,\gamma\,(\gamma+1)}\,x^2+\dots$$

by $F\left\{\alpha,\ \beta,\ \gamma,\ x\right\}$, it is well known that $F\left\{\alpha,\ \beta,\ \gamma,\ 1\right\}$ is convergent if $\gamma > \alpha+\beta$. Now the given series $= F\left\{m,\ \frac{q}{r},\ \frac{p}{r},\ 1\right\}$, and is therefore convergent if $p > q+mr$.

7244. (D. EDWARDES.)—The circles of curvature at three points of an ellipse meet in a point P on the curve. Prove that (1) the normals at these three points meet on the normal drawn at the other extremity of the diameter through P; and (2) the locus of their point of intersection for different positions of P is $4\,(a^2x^2+b^2y^2) = (a^2-b^2)^2$.

Solution by the Rev. T. C. SIMMONS, M.A.

Let α be the eccentric angle of P′, the other extremity of the diameter, that of P being of course $\pi+\alpha$, and let $\beta,\ \gamma,\ \delta$ be the eccentric angles of the points of contact of the circles of curvature, then

$$x/a\cos\tfrac{1}{2}\,(\pi+\alpha+\beta)+y/b\sin\tfrac{1}{2}\,(\pi+\alpha+\beta) = \cos\tfrac{1}{2}\,(\pi+\alpha+\beta),$$

or $$x/a\sin\tfrac{1}{2}\,(\alpha+\beta)-y/b\cos\tfrac{1}{2}\,(\alpha+\beta) = \sin\tfrac{1}{2}\,(\alpha-\beta)\ \dots\dots\dots\dots\dots (1),$$

the chord joining P with β, and $x/a\cos\beta+y/b\sin\beta = 1$ .... (2);

the tangent at β, must make equal angles with the axes, therefore

$$\sin\tfrac{1}{2}\,(\alpha+\beta)\sin\beta-\cos\tfrac{1}{2}\,(\alpha+\beta)\cos\beta = 0,$$

or $$\alpha+3\beta = (2m+1)\,\pi.$$

Similarly, $$\alpha+3\gamma = (2n+1)\,\pi,\quad \alpha+3\delta = (2p+1)\,\pi;$$

therefore $\alpha+\beta+\gamma+\delta$ is an odd multiple of π, or the normals at $\alpha,\ \beta,\ \gamma,\ \delta$ are concurrent.

Again, if $\alpha,\ \beta,\ \gamma,\ \delta$ occur in this order, since their differences must be even multiples of $\frac{1}{3}\pi$, it is evident that, when they are unequal, $\beta-\gamma$, $\gamma-\delta$ each $= \frac{2}{3}\pi$; in other words, $\beta,\ \gamma,\ \delta$ are at the vertices of a maximum triangle in the ellipse. Consider now the normals at $\beta,\ \gamma$: they are

$$2ax\sin\beta-2by\cos\beta = (a^2-b^2)\sin 2\beta\ \dots\dots\dots\dots\dots\dots (3),$$

and $$2ax\sin(\beta+\tfrac{2}{3}\pi)-2by\cos(\beta+\tfrac{2}{3}\pi) = (a^2-b^2)\sin(2\beta+\tfrac{4}{3}\pi),$$

which reduces to $$2ax\,(\sqrt{3}\cos\beta-\sin\beta)+2by\,(\sqrt{3}\sin\beta+\cos\beta)$$

$$= -(a^2-b^2)(\sin 2\beta+\sqrt{3}\cos 2\beta)\dots\dots (4),$$

whence, by (3) + (4), $2ax\cos\beta+2by\sin\beta = -(a^2-b^2)\cos 2\beta$ .... (5).

Squaring and adding (3) and (5), we obtain for the locus of the intersection of the normals $\quad 4\,(a^2x^2+b^2y^2) = (a^2-b^2)^2$.

9418. (Professor Sylvester, F.R.S.)—If p, i, j are each prime numbers, and $1+p+p^2+\ldots+p^{i-1} = q^j$, prove that j is a divisor of $q-i$. Example: $1+3+3^2+3^3+3^4 = 11^2$, and 2 is a divisor of $11-5$.

Solution by W. S. Foster.

$$1+p+p^2+\ldots+p^{i-1} = q^j;$$

therefore $$q^{j-1}+q^{j-2}+\ldots+q+1 = \frac{p}{q-1}(1+p+p^2+\ldots+p^{i-2}),$$

and p is a prime number; hence, as in Question 9381, j divides $p-1$, or $j = p$ and divides $q-1$. In the first case, $p = Aj+1$; therefore

$$1+(Aj+1)+(Aj+1)^2+\ldots+(Aj+1)^{i-1} = q^j;$$

therefore $\quad i+Bj = q^j$; therefore $Bj = q^j-q+q-i$;

and, since j is a prime number, q^j-q is divisible by j, therefore $q-i$ is divisible by j. In the second case, we should have

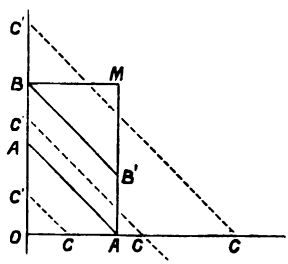
$$1+p+p^2+\ldots = (Cp+1)^p = 1+C_1p^2+C_2p^3+\ldots,$$

which is impossible if p is a prime number.

9423. (Professor Neuberg).—On casse, au hasard, deux barres de longueurs a et b, chacune en deux morceaux. Quelle est la probabilité qu'un morceau de la première barre et un morceau de la seconde, étant juxtaposés, donnent une longueur moindre que c?

Solutions by (1) Professor De Wachter, (2) Professor Schoute.

1. Assume two rectangular axes, OX, OY; take OA $= a$, and OB $= b$, and describe the rectangle OAMB. From any interior point if perpendiculars be drawn to OA and OB, they may represent, in one of the possible combinations, the parts of a and b to be added together.



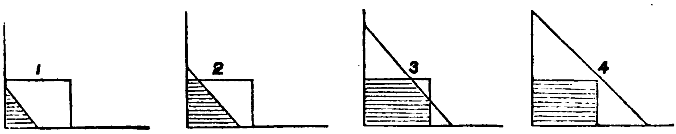
The amount of possible chances will be represented by the area OAMB and measured by ab. On OX and OY, respectively, make OC $=$ OC$'$ $=$ given length c, and join CC$'$. The sum of the distances of any point in CC$'$ from the axes is $= c$. First of all, we must have $a+b > c$, if CC$'$ is to cut off a part from OAMB. The required probability P is the ratio of the area limited by the axes to the rectangle OAMB. Draw AA$'$ and BB$'$ parallel to CC$'$. Assuming $a+b > c$ and $b > a$, three cases are possible.

(1) $b > a > c$. CC$'$ falls between AA$'$ and O, and $P = c^2/2ab$.

(2) $b > c > a$. CC$'$ lies between AA$'$ and BB$'$, and $P = (2c-a)/2b$.

(3) $c > b > a$. CC$'$ falls between M and BB$'$, and $P = 1-(a+b-c)^2/2ab$.

2. *Otherwise* :—Si l'on représente les longueurs des deux morceaux qu'on réunit par x et y, on a les conditions générales $0 < 2x < a$ et $0 < 2y < b$, tandis que les cas favorables sont soumis à la troisième condition $x+y < c$. En considérant x et y comme les coordonnées rectangulaires d'un point dans le plan, on trouve pour la probabilité en question d'après les figures suivantes :



pour $c < \frac{1}{2}b$ (Fig. 1) .... $P = 4c^2/2ab$,

pour $\frac{1}{2}b < c < \frac{1}{2}a$ (Fig. 2) .... $P = [4c^2-(2c-b)^2]/2ab$,

pour $\frac{1}{2}a < c < \frac{1}{2}(a+b)$ (Fig. 3) ...... $P = [4c^2-(2c-b)^2-(2c-a)^2]/2ab$,

pour $\frac{1}{2}(a+b) < c$ (Fig. 4) .... $P = 1$.

On a supposé $a > b$; quand $a = b$, le cas de Fig. 2 disparaît.

9406. (W. J. Barton, M.A.)—Show that, if $R = 2r$, the triangle is equilateral, *without* employing the expression for the distance between the centres.

Solution by Professor Emmerich, Ph.D.

If $R = 2r$, the incircle is equal to the nine-point circle. After Feuerbach's theorem, the incircle of each triangle is touched by the nine-point circle, and it may easily be seen that the incircle lies entirely inside the nine-point circle ; for, if the bisector AD of the angle A meets the circumcircle at E, the incentre T lies between A and E ; therefore the projection of the point T upon BC lies between the projections of the points A, E ; that is to say, the first projection, which is a point of the incircle, lies inside the nine-pcint circle. Therefore, if $R = 2r$, the two circles coincide. Hence the sides are touched in their mid-points by the incircle, etc.

1898 & 4043. (Hugh MacColl, B.A.)—Find the number and situation of the real roots, giving a near approximation to each, of

$$x^4+4\cdot37162x^3-24\cdot9642358761x^2+34\cdot129226840859882x$$
$$-14\cdot6344200781857045 2204 = 0.$$

Solution by D. Biddle.

The ordinary 7-figure Tables of Logarithms do not permit of a near approach to accuracy in this investigation. But we can proceed thus :—

Rendering the equation $x^4+ax^3-bx^2+cx-d = 0$ ....(1),

form another equation thus: $(x^2+yx-z)(x^2-ux+v)=0$.................(2), in which $y-u=a$, $uy+z-v=b$, $uz+vy=c$, and $vz=d$, whence $v=d/z$, $y=(az^2+cz)/(z^2+d)$, and $u=(cz-ad)/(z^2+d)$. Then find any one real root of x from the original equation. A very good method of effecting this, in equations like the present, is to separately record the portions of x as they are found, and amend the coefficients a, b, c, d, so as to form an equation similar in kind, but of which the *remainder* of x is the unknown. Thus, let $A_n \equiv$ the last portion of x found, consisting by preference of only one figure, and $d_{n+1} \equiv$ the error resulting from it; also let $h \equiv$ the addition necessary to raise $(A_1+A_2+\ldots A_n)$ to x. Then

$$A_n^4+a_nA_n^3-b_nA_n^2+c_nA_n-d_n=-d_{n+1}\ \ldots\ldots\ldots\ldots\ldots(3),$$

$$(A_n+h)^4+a_n(A_n+h)^3-b_n(A_n+h)^2+c(A_n+h)-d_n=0\ \ldots\ldots(4);$$

expanding (4) and subtracting from it (3), we have

$$h^4+(a_n+4A_n)h^3-(b_n-6A_n^2-3A_na_n)h^2$$
$$+(c+4A_n^3+3A_n^2a_n-2A_nb_n)h-d_{n+1}=0\ \ldots\ldots\ldots(5),$$

which may be rendered $h^4+a_{n+1}h^3-b_{n+1}h^2+c_{n+1}h-d_{n+1}=0$ ....(6).

This is a condensed form of the old method of extracting roots of the fourth power, and is scarcely to be surpassed for equations such as the present until full logarithmic tables to 24 places of decimals are provided.

A	a	b
1·000,000,0	4·371,620	24·964,235,876,1
·100,000,0	8·371,620	5·849,375,876,1
·100,000,0	8·771,620	3·277,889,876,1
·010,000,0	9·171,620	0·586,403,876,1
·006,000,0	9·211,620	0·310,655,276,1
·001,000,0	9·235,620	0·144,630,116,1
·000,900,0	9·239,620	0·116,929,256,1
·000,080,0	9·243,220	0·091,981,472,1
·000,001,0	9·243,540	0·089,763,060,9
·000,000,9	9·243,544	0·089,735,338,4
1·217,981,9 = x_1		

c	d
34·129,226,840,859,882	14·634,420,078,185,704,522,04
1·315,615,088,659,882	0·097,809,113,425,822,522,04
0·400,888,513,439,882	0·016,269,743,320,834,322,04
0·012,459,138,219,882	0·000,088,170,737,846,122,04
0·003,486,546,697,882	0·000,013,038,123,257,302,04
0·001,617,538,344,682	0·000,001,311,427,089,610,04
0·001,355,988,972,482	0·000,001,130,953,285,928,04
0·001,147,764,455,162	0·000,000,093,452,567,496,24
0·001,133,244,891,498	0·000,000,002,215,359,935,12
0·001,133,065,393,107	0·000,000,001,082,204,797,44
The final remainder =	0·000,000,000,062,445,936,90

Having thus found that $x_1=1·2179819$, we revert to (2), whence we obtain $x^2+yx-z=0$, and $x^2-ux+v=0$.................(7, 8).

Expressing (7) in the terms given under (2), we have

$$z^3-(ax+x^2)z^2-(cx-d)z-dx^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

or $$z^3 - 6{\cdot}80803394456561z^2 - 26{\cdot}93434047497581z$$
$$-21{\cdot}70986819346878 = 0 \ldots\ldots\ldots(10),$$
which has two real roots, namely $z_1 = 9{\cdot}7874906$, $z_2 = -1{\cdot}4846695$.

From (7), (8), $$x = \pm\left\{z + \tfrac{1}{4}\left(\frac{az^2+cz}{z^2+d}\right)^2\right\}^{\frac{1}{2}} - \tfrac{1}{2}\left(\frac{az^2+cz}{z^2+d}\right) \ldots\ldots\ldots\ldots(11),$$
$$x = \pm\left\{\tfrac{1}{4}\left(\frac{cz-ad}{z^2+d}\right)^2 - \frac{d}{z}\right\}^{\frac{1}{2}} + \tfrac{1}{2}\left(\frac{cz-ad}{z^2+d}\right) \ldots\ldots\ldots\ldots(12).$$

Taking z_1, we obtain, from (11), $x_1 = 1{\cdot}2179819$ and $x_2 = -8{\cdot}0352616$, and from (12) the two imaginary roots $1{\cdot}2227875 \pm (-{\cdot}0000075)^{\frac{1}{2}}$. Taking z_2, we obtain from (11) two more imaginary roots,
$$1{\cdot}2184609 \pm (-{\cdot}0000225)^{\frac{1}{2}},$$
and from (12), $x_2 = -8{\cdot}0352616$ and $x_3 = 1{\cdot}2266901$, which is quite distinct from x_1.

For the service of those who may wish to carry the investigation to a great degree of nicety, the logarithms of a, b, c, d are here given to 24 places of decimals, by aid of PETER GRAY's admirable method:—

$$\log a = 0{\cdot}640642404175296337879050,$$
$$\log b = 1{\cdot}397318277386039907465102,$$
$$\log c = 1{\cdot}533126449938424905356617,$$
$$\log d = 1{\cdot}165375517215043330021209.$$

9392. (Professor GENESE, M.A.)—If the tangent at any point P of a folium of Descartes meet the tangents at the node in X, Y, and the curve again at Q, then prove that $\dfrac{1}{PX} + \dfrac{1}{PY} = \dfrac{3}{PQ}$.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

The equation of the curve being $x^3 + y^3 = axy$, any point P may be taken, $x = \dfrac{a}{1+t^3}$, $y = \dfrac{at^2}{1+t^3}$; and if any transversal $px + qy = a$ meet this in three points, the values of t at the three points will be given by $pt + qt^2 = 1 + t^3$, if $t_1t_2t_3$ be the three $t_1t_2t_3 = -1$. Hence, if t, t' be the values at P, Q, $t^2t' = -1$; also the equation of the tangent at P will be $x(2t - t^4) + y(2t^3 - 1) = at^2$, and if O be the origin, and PR the harmonic mean between PX, PY, the equation of OR will be $y = -tx$. Now, the coordinates of a point dividing PQ in the ratio $m : l$ will be in the ratio
$$\frac{lt^2}{1+t^3} + \frac{mt'^2}{1+t'^3} : \frac{lt}{1+t^3} + \frac{mt'}{1+t'^3}, \text{ or } \frac{lt^2}{1+t^3} + \frac{mt^2}{t^6-1} : \frac{lt}{1+t^3} - \frac{mt^4}{t^6-1};$$
which $= t\left\{l(t^3-1) + m\right\} : l(t^3-1) - mt^3$; if $m = 2l'$, this ratio is $-t : 1$.

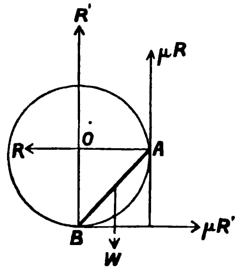
Hence $$\text{PR} = \tfrac{2}{3}\text{PQ}, \text{ and } \frac{1}{PX} + \frac{1}{PY} = \frac{2}{PR} = \frac{3}{PQ}.$$

The pedal of the parabola $y^2 = 4ax$ with respect to the point $(-3a, 0)$ is

an orthogonal projection of the folium, and hence the property will be true for this pedal, but I find on trial it is not true for any other pedal of the parabola. In this pedal, the circular points are inflexions; as, in the folium, the three points at infinity are inflexions.

8503. (N'IMPORTE.)—A rod of length $a\sqrt{2}$ rests in equilibrium in a vertical plane within a rough sphere of radius a, one extremity of the rod being at the lowest point of the sphere; show that the coefficient of friction is $\sqrt{2}-1$.

Solution by GEORGE GOLDTHORPE STORR, M.A.



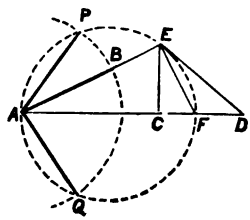
Let AB be the rod, O the centre of the sphere, μ the coefficient of friction, and W the weight of the rod; then, resolving vertically and horizontally, and taking moments about B, we have

$$R' + \mu R = W, \quad R = \mu R',$$
$$\mu Ra + Ra = W\tfrac{1}{2}a,$$

whence $\mu = \sqrt{2}-1 = \cdot 4141$ nearly.

9436. (W. GALLATLY, M.A.)—AB is a mirror swinging on a hinge at A. At C is a candle flame, and at D an observer; the line ACD being perpendicular to the axis of the mirror. Find geometrically the position of the mirror, when the observer at D sees the image of the flame on the point of disappearing.

Solution by Professor SCHOUTE.



Let the ray emitted by C, that would be observed at D when the mirror in the arbitrary position AB was long enough, fall on the produced mirror in E. Then AE is the external bisector of angle E of triangle CDE. Therefore EC/ED = AC/AD. This proves that the locus of E is the circumcircle of the triangle AEF, EF being the internal bisector of angle E. So the two limiting positions of the mirror are given by the joins of A with either of the points P, Q common to this circle and the circle with A as centre and AB as radius.

6911. (W. R. WESTROPP ROBERTS, M.A.)—Let H and H′ be the Hessians of two binary cubics respectively, Θ their intermediate covariant; then, using the notation of SALMON, prove that

$$9\Theta^2 - 36HH' = 6PJ + H(6J).$$

Solution by D. EDWARDES.

With the notation of SALMON's *Higher Algebra*, Art. 216, the sources of H, H′, Θ, H (6J) are respectively,

$$ac-b^2,\quad a'c'-b'^2,\quad ac'+ca'-2bb',\quad 36\,(a_0a_2-a_1^2).$$

Now
$$4\,(ac-b^2)\,(a'c'-b'^2)-(ac'+a'c-2bb')^2$$
$$= 4\,(bc')\,(ab')-(ca')^2 = 4\,(a_2-\tfrac{1}{8}P)\,a_0-4a_1^2;$$

hence, for the corresponding covariants, we have the stated result.

9369. (W. J. C. SHARP, M.A.)—Prove, from the theory of combinations, (1) that $1+\dfrac{m}{1}\cdot\dfrac{n}{1}+\dfrac{m\,(m-1)}{1\,.\,2}\cdot\dfrac{n\,(n-1)}{1\,.\,2}+\ldots=\dfrac{(m+n)\,!}{m\,!\;n\,!}$ must be true; and (2) deduce that, if (m) be a prime greater than (n), $(m+n)\,!-m\,!\;n\,!$ and $\dfrac{(m+n)\,!}{m\,!}$ are respective multiples of (m^2), (m).

Solution by Professor IGNACIO BEYENS.

1. Designant par C_m^n le nombre des combinaisons de (m) lettres prises (n) à (n), on aura d'abord $\dfrac{(m+n)\,!}{m\,!\;n\,!}=C_{m+n}^n$ et aussi,

$$C_{m+n}^n = C_m^m\,.\,C^n + C_m^1\,.\,C_n^{n-1} + C_m^2\,.\,C_n^{n-2} + \ldots + C_m^{n-1}\,C_n^1,$$

mais
$$C_n^{n-1} = C_n^1,\quad C_n^{n-2} = C_n,$$

donc
$$1+\frac{m}{1}\cdot\frac{n}{1}+\frac{m\,(m-1)}{1\,.\,2}\cdot\frac{n\,(n-1)}{1\,.\,2}+\ldots=\frac{(m+n)\,!}{m\,!\;n\,!}.$$

2. De cette équation on déduira

$$\left(\frac{m}{1}\cdot\frac{n}{1}+\frac{m\,(m-1)}{1\,.\,2}\cdot\frac{n\,(n-1)}{1\,.\,2}+\ldots\right)m\,!\;n\,! = (m+n)\,!-m\,!\;n\,!,$$

et si (m) est un nombre premier plus grand que (n) est évident que le premier membre est toujours multiple de $m\,.\,m=m^2$, donc

$$(m+n)\,!-m\,!\;n\,! = M\,.\,(m^2)\,;$$

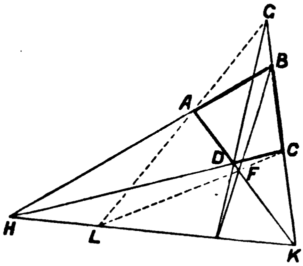
et de la même relation on a

$$m\,!\left(1+\frac{m}{1}\cdot\frac{n}{1}+\frac{m\,(m-1)}{1\,.\,2}\cdot\frac{n\,(n-1)}{1\,.\,2}+\ldots\right)=\frac{(m+n)\,!}{n\,!},$$

donc $\dfrac{(m+n)\,!}{n\,!}$ est aussi multiple de m.

9149. (Charlotte A. Scott, B.Sc.)—If ABCD be a quadrilateral, in which the sides BA, CD meet towards A and D in H, and the sides BC, AD meet towards C and D in K; and if from a point L in HK, LAG, LFC be drawn meeting BC in G and AD in F, respectively; show that BF and GD meet in HK.

Solution by D. O. S. Davies, M.A.; G. G. Morrice, M.A.; *and others.*



The triangles ABG, CDF are coplanar. Hence the diagonals of the quadrilateral ABCD, AGCF pass through same point. Therefore the triangles AGD, CFB are copolar, and therefore coplanar. Therefore GD, BF intersect in HK.

9414. (R. W. D. Christie.)—If $2^p - 1$ is a prime, show that p is also prime. [Better thus:—What prime p will make $2^p - 1$ a prime?]

Solution by Professor Ignacio Beyens; D. Watson; *and others.*

Si (p) ne fusse pas nombre premier, supposons $p = np'$; alors on aurait $2^p - 1 = 2^{np'} - 1 =$ multiple de $(2^n - 1) =$ multiple $(2^{p'} - 1)$, et $2^p - 1$ ne serait pas de nombre premier, ce qui est contraire à l'énoncé. [Prof. Beyens adds the following generalization:—"Si $(n+1)^p - n^p$ est un nombre premier, l'exposant (p) est aussi un nombre premier, parce que si (p) n'était pas de nombre premier, soit $p = mp'$; alors $(n+1)^p - n^p = (n+1)^{mp'} - n^{mp'}$ serait un multiple de $(n+1)^{p'} - n^{p'})$ et de $(n+1)^m - n^m$." Mr. Christie remarks that the solution shows that p is a prime is a *necessary* condition, but not that it is a sufficient one, which is the real problem in perfect numbers.]

9164. (Professor Nilkantha Sarkar, M.A.)—Prove that

$$\frac{2}{\pi}\int_0^\pi e^{c\cos x}\sin(c\sin x)\sin nx\,dx = \frac{c^n}{n!}.$$

Solution by D. Edwardes.

Let
$$S_c = 1 + c\cos x + \frac{c^2}{2!}\cos 2x + \frac{c^3}{3!}\cos 3x + \&c.,$$

$$S_s = c\sin x + \frac{c^2}{2!}\sin 2x + \&c., \text{ and } j^2 = -1.$$

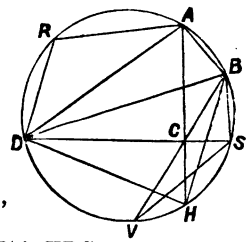
Then
$$S_c + jS_s = \epsilon^{c(\cos x + j\sin x)} = \epsilon^{c\cos x}\left\{\cos(c\sin x) + j\sin(c\sin x)\right\};$$

therefore
$$S_s = \epsilon^{c\cos x}\sin(c\sin x).$$

Hence, substituting the series for $\epsilon^{c\cos x}\sin(c\sin x)$, the integral reduces to
$$\frac{2}{\pi}\cdot\frac{c^n}{n!}\int_0^\pi \sin^2 nx\,dx = \frac{c^n}{n!}.$$

9427. (Professor GENESE, M.A.)—If A, B, C, D be points in a plane, prove that
$$\frac{\mathrm{BC}\,.\,\mathrm{AD}}{\sin(\mathrm{BAC}-\mathrm{BDC})} = \frac{\mathrm{CA}\,.\,\mathrm{BD}}{\sin(\mathrm{CBA}-\mathrm{CDA})} = \frac{\mathrm{AB}\,.\,\mathrm{CD}}{\sin(\mathrm{ACB}-\mathrm{ADB})},$$
where any angle BAC means the angle through which AC must be turned *in the positive sense* to coincide with AB.

Solution by Professors W. P. CASEY, MATZ; *and others.*



Describe a circle about ABD, produce AC to H, and join DH, BH.

Then $\angle\mathrm{HDC} = \angle\mathrm{BAC} - \angle\mathrm{BDC}$,
and $\angle\mathrm{CBH} = \angle\mathrm{ACB} - \angle\mathrm{ADB}$,
but $\mathrm{CD/CH} = \sin\mathrm{DHC}/\sin\mathrm{HDC}$,
and $\mathrm{CH/BC} = \sin\mathrm{CBH}/\sin\mathrm{CHB}$;
thus
$\mathrm{CD/BC} = \sin\mathrm{DHC}\sin\mathrm{CBH}/\sin\mathrm{HDC}\,.\,\sin\mathrm{CHB}$,
but $\sin\mathrm{DHC}/\sin\mathrm{CHB} = \mathrm{AD/AB}$,
therefore
$$\mathrm{CD}\,.\,\mathrm{AB}/\mathrm{BC}\,.\,\mathrm{AD} = \sin\mathrm{CBH}/\sin\mathrm{HDC}$$
$$= \sin(\mathrm{ACB}-\mathrm{ADB})/\sin(\mathrm{BAC}-\mathrm{BDC}).$$

Again, make $\angle\mathrm{CDR} = \angle\mathrm{CBA}$, and therefore $\angle\mathrm{ADR} = \angle\mathrm{CBA} - \angle\mathrm{CDA}$. Produce BC, DC to V and S. Join VS, SB, and AR, and from the similar triangles of this figure, after a little reduction, we get
$$\mathrm{BC}\,.\,\mathrm{AD}/\mathrm{AC}\,.\,\mathrm{BD} = \mathrm{AD}\,.\,\mathrm{CD}/\mathrm{CH}\,.\,\mathrm{AR} = \sin\mathrm{HDC}/\sin\mathrm{ADR},$$
or
$$\mathrm{BC}\,.\,\mathrm{AD}/\sin(\mathrm{BAC}-\mathrm{BDC}) = \mathrm{AC}\,.\,\mathrm{BD}/\sin(\mathrm{CBA}-\mathrm{CDA}).$$

9391. (Professor SATIS CHANDRA RAY, M.A.)—If the diagonals of a cyclic quadrilateral ABCD intersect in O; and if AB $= a$, BC $= b$, CD $= c$, DA $= d$, $\angle\mathrm{AOD} = \mathrm{ADB}$; prove that
$$(bc+ad)(cd+ab)/(ac+bd) = a^2.$$

Solution by J. YOUNG, M.A.; G. G. STORR, M.A.; *and others.*

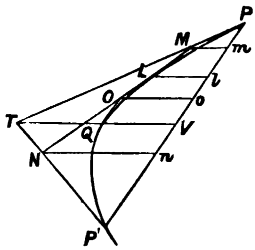
For all cyclic quadrilaterals $\dfrac{bc+ad}{\mathrm{AC}} = \dfrac{cd+ab}{\mathrm{BD}} = \dfrac{ad}{\mathrm{AO}}$,

and in this case $d = \mathrm{AO}$; hence $\dfrac{(bc+ad)(cd+ab)}{\mathrm{AC}\,.\,\mathrm{BD}} = a^2$,

and $\mathrm{AC}\,.\,\mathrm{BD} = ac+bd$.

9380. (SARAH MARKS, B.Sc.)—Tangents are drawn to a parabola from a point T; a third tangent meets these in MN; prove that the polar of the mid-point of MN and the diameter through T meet on the parabola.

Solution by C. E. WILLIAMS, M.A.; R. KNOWLES, B.A.; *and others.*



The diameter through T is TQV, QO the tangent at Q, meeting MN at O; Mm, Ll, Oo, Nn diameters; then m, n, o, V, are the mid-points of Pl, P$'l$, lV, PP$'$; therefore o, O are the mid-points of mn, MN; hence the polar of O, the mid-point of MN, is QL.

8826. (Professor SIRCOM, M.A. Suggested by Question 2845.)—Show that $1+\dfrac{2}{3}x^2+\dfrac{2\,.\,4}{3\,.\,5}x^4+\dfrac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}x^6+\ldots = \dfrac{\sin^{-1}x}{x(1-x^2)^{\frac12}}$.

Solution by the PROPOSER; Professor CHAKRAVARTI, M.A.; *and others.*

The differential equation satisfied by the given series is found by the usual methods to be $(x-x^3)\,dy/dx+(1-2x^2)\,y = 1$,

of which the solution is $x(1-x^2)^{\frac12}y = \sin^{-1}x + \mathrm{C}$,

which is satisfied by the given series if $\mathrm{C} = 0$.

9384. (Professor BORDAGE.)—Show that the roots of the equation $(x+2)^2+2(x+2)\sqrt{x}-2x-3\sqrt{x}-46 = 0$ are 9, 4, $\frac12\left\{-13\pm3(-3^{\frac12})\right\}$.

Solution by ELEANOR ROBINSON; G. G. STORR, M.A.; *and others.*

This is $(x+x^{\frac12}+2)^2-3(x+x^{\frac12}+2) = 40$, whence $x+x^{\frac12}+2 = 8$ or -5, $x^{\frac12} = -3,\ +2,\ \frac12\left\{-1\pm3(-3)^{\frac12}\right\}$, and $x = 9,\ 4,\ \frac12\left\{-13\mp3(-3)^{\frac12}\right\}$.

9371. (J. BRILL, M.A.)—Prove that in any triangle, n being a positive integer,

$$a^n \cos nB + b^n \cos nA$$

$$= c^n - nabc^{n-2}\cos(A-B) + \frac{n(n-3)}{2!}a^2b^2c^{n-4}\cos 2(A-B)$$

$$-\frac{n(n-4)(n-5)}{3!}a^3b^3c^{n-6}\cos 3(A-B)$$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}a^4b^4c^{n-8}\cos 4(A-B) - \&c.$$

Solution by Professor SIRCOM, M.A.; H. FORTEY, M.A.; *and others.*

We have $a\cos B + b\cos A = c$, $a\sin B - b\sin A = 0$, whence

$$ae^{\iota B} + be^{-\iota A} = c, \quad ae^{-\iota B} + be^{\iota A} = c.$$

Now the sum of the n^{th} powers of the roots of $x^2 - px + q = 0$ (TODHUNTER's *Theory of Equations*, p. 182) is

$$p^n - np^{n-2}q + \frac{n(n-3)}{2!}p^{n-4}q^2 - \frac{n(n-4)(n-5)}{3!}p^{n-6}q^3 + \&c.,$$

and $ae^{\iota B}$, $be^{-\iota A}$ are roots of $x^2 - cx + abe^{-\iota(A-B)} = 0$, $ae^{-\iota B}$, $be^{\iota A}$ are roots of $x^2 - cx + abe^{\iota(A-B)} = 0$, whence substituting for p and q from each of these equations, and adding, we obtain the required result.

[The same method applies to Quest. 8290, which is otherwise solved on p. 96 of Vol. XLV.]

9319 & 9364. (Professor BHATTACHARYYA.)—(9319.) Show that

$$\frac{(2m+1)(2m+3)\dots(2m+2r-1)}{r!} + \frac{(2m+1)(2m+3)\dots(2m+2r-3)}{(r-1)!}\cdot\frac{2n-1}{1}$$

$$+\frac{(2m+1)(2m+3)\dots(2m+2r-5)}{(r-2)!}\cdot\frac{(2n-1)(2n+1)}{2!}+\dots$$

$$=\frac{(m+n+r-1)!}{(m+n-1)!\,r!}2^r.$$

(9364.) (W. J. GREENSTREET, B.A.)—If q is any positive integer, prove that $$\frac{2^q}{q+1} = 1 + \tfrac{1}{3}\frac{q(q-1)}{2!} + \tfrac{1}{5}\frac{q(q-1)(q-2)(q-3)}{4!} + \dots$$

Solution by R. F. DAVIS, M.A.; W. J. BARTON, M.A.; *and others.*

(9319.) This identity follows from equating the coefficients of x^r in the expansion of $(1-x)$ to the power $-(m+n)$ and in the product of the expansions of the same binomial to the powers $-\frac{1}{2}(2m+1)$ and $-\frac{1}{2}(2n-1)$.

(9364.) This identity follows from the fact that the sum of the odd coefficients in the expansion of $(1+x)$ to the power $(q+1) = \frac{1}{2}\cdot 2^{q+1} = 2^q$.

9325. (S. TEBAY, B.A.)—A, B, C are the dihedral angles at the base of a tetrahedron; X, Y, Z the respective opposites; show that, if

$$T_1 = (1-\cos^2 B - \cos^2 C - \cos^2 X - 2\cos B\cos C\cos X)^{\frac{1}{2}},$$

with similar expressions (denoted by T_2, T_3, T_4) for the other solid angles,

$$T_2T_3\cos X + T_3T_1\cos Y + T_1T_2\cos Z = 1 - \cos^2 A - \cos^2 B - \cos^2 C$$
$$-\cos B\cos C\cos X - \cos C\cos A\cos Y - \cos A\cos B\cos Z + \cos X\cos Y\cos Z.$$

Solution by D. EDWARDES; Prince de POLIGNAC; *and others.*

Denoting the areas of the faces by P, Q, R, S, we have by projection the equations $\quad P\cos A + Q\cos B + R\cos C - S = 0,$

$$-P + Q\cos Z + R\cos Y + S\cos A = 0,\quad P\cos Z - Q + R\cos X + S\cos B = 0,$$
$$P\cos Y + Q\cos X - R + S\cos C = 0,$$

therefore $\quad P\begin{vmatrix}\cos A, & \cos B, & \cos C\\ \cos Z, & -1, & \cos X\\ \cos Y, & \cos X, & -1\end{vmatrix} = S\begin{vmatrix}1, & \cos B, & \cos C\\ -\cos B, & -1, & \cos X\\ -\cos C, & \cos X, & -1\end{vmatrix}.$

Therefore, squaring, $P^2\,|\,\&c.\,|^2 = S^2\,|\,\&c.\,|^2$. But we have the symmetrical determinant

$$\begin{vmatrix}-1, & \cos A, & \cos B, & \cos C\\ \cos A, & -1, & \cos Z, & \cos Y\\ \cos B, & \cos Z, & -1, & \cos X\\ \cos C, & \cos Y, & \cos X, & -1\end{vmatrix} = 0,\quad \therefore \begin{vmatrix}\cos A, & \cos B, & \cos C\\ \cos Z, & -1, & \cos X\\ \cos Y, & \cos X, & -1\end{vmatrix}^2$$
$$= \begin{vmatrix}-1, & \cos B, & \cos C\\ \cos B, & -1, & \cos X\\ \cos C, & \cos X, & -1\end{vmatrix} \times \begin{vmatrix}-1, & \cos Z, & \cos Y\\ \cos Z, & -1, & \cos X\\ \cos Y, & \cos X, & -1\end{vmatrix},$$

therefore $\quad \dfrac{P^2}{\begin{vmatrix}1, & \cos B, & \cos C\\ -\cos B, & -1, & \cos X\\ -\cos C, & \cos X, & -1\end{vmatrix}} = \dfrac{S^2}{\begin{vmatrix}-1, & \cos Z, & \cos Y\\ \cos Z, & -1, & \cos X\\ \cos Y, & \cos X, & -1\end{vmatrix}},$

i.e., $P/T_1 = S/T$, where $T = 1 - \cos^2 X - \cos^2 Y - \cos^2 Z - 2\cos X\cos Y\cos Z$;

therefore $\quad P/T_1 = Q/T_2 = R/T_3 = S/T.$

Now, from above, $\;PQ\cos Z + PR\cos Y = P^2 - SP\cos A,$

and two similar equations. Adding these,

$$2\,(PQ\cos Z + QR\cos X + RP\cos Y) = P^2 + Q^2 + R^2 - S^2,$$

and substituting for P, Q, R, S the quantities T_1, T_2, &c., to which they are proportional, and reducing on right side, we have the required result.

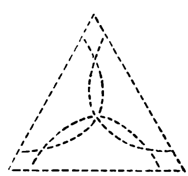
[If we form similar relations for the other three faces, and add all four together, we obtain

$$T_2T_3\cos X + T_3T_1\cos Y + T_1T_2\cos Z + T_1T_4\cos A + T_2T_4\cos B + T_3T_4\cos C$$
$$= 2 - \cos^2 A - \cos^2 B - \cos^2 C - \cos^2 X - \cos^2 Y - \cos^2 Z - \cos B\cos C\cos X$$
$$-\cos C\cos A\cos Y - \cos A\cos B\cos Z - \cos X\cos Y\cos Z.]$$

9200. (Professor NEUBERG.)—On casse, au hasard, une barre, de longueur $3a$, en trois morceaux. Démontrer que la probabilité que le produit des longueurs de deux quelconques des morceaux soit moindre que a^2 est :

$$\tfrac{4}{3}\log_e\left[\tfrac{1}{2}(3+\sqrt{5})\right]+2-\sqrt{5} = 0{\cdot}123 \text{ (à très-peu près).}$$

Solution by Professor F. X. DE WACHTER.



Dans le triangle équilatéral de hauteur $3a$, déterminons le lieu des points dont les distances à deux des côtés ont pour produit a^2. Ce lieu se compose de trois arcs hyperboliques ayant le centre du triangle pour sommet commun et les deux côtés correspondants pour asymptotes. Il est aisé de voir que l'espace favorable à l'événement considéré se compose de 3 quadrilatères mixtilignes égaux, situés dans les coins du triangle. Donc la probabilité cherchée a la valeur donnée.

5440. (R. RAWSON.)—Prove that the general solution of the equation is

$$u = c_3\int_\beta^\alpha \epsilon^{x_1\phi(z)-c_2'\phi(z)}\,.\,\phi(z)^{c_1}\phi'(z)\,dz + c \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d^2u}{dx^2}+\left\{\frac{c_1+2}{x_1}\left(\frac{dx_1}{dx}\right)^2-\frac{d^2x_1}{dx^2}\right\}\frac{du}{dx}\,\frac{dx}{dx_1}+\frac{c_2}{x_1}\left(\frac{dx_1}{dx}\right)^2$$

$$=\left\{\frac{c_1+2}{x_1}\left(\frac{dx_1}{dx}\right)^2-\frac{d^2x_1}{dx^2}\right\}\mathrm{N}+\frac{d\mathrm{N}}{dx}+\mathrm{M}+\frac{cc_2}{x_1}\left(\frac{dx_1}{dx}\right)^2$$

$$+\frac{c_3}{x_1}\left(\frac{dx_1}{dx}\right)^2\left\{\epsilon^{x_1\phi(\alpha)-c_2'\phi(\alpha)}\,.\,\phi(\alpha)^{c_1+2}-\epsilon^{x_1\phi(\beta)-c_2'\phi(\beta)}\,.\,\phi(\beta)^{c_1+2}\right\},$$

where α, β, x_1 are given functions of x, and

$$\mathrm{N}=c_3\left\{\frac{d\alpha}{dx}\,\epsilon^{x_1\phi(\alpha)-c_2'\phi(\alpha)}\,.\,\phi(\alpha)^{c_1}\phi'(\alpha)-\frac{d\beta}{dx}\,\epsilon^{x_1\phi(\beta)-c_2'\phi(\beta)}\,.\,\phi(\beta)^{c_1}\phi'(\beta)\right\},$$

$$\mathrm{M}=c_3\frac{dx_1}{dx}\left\{\frac{d\alpha}{dx}\,\epsilon^{x_1\phi(\alpha)-c_2'\phi(\alpha)}\,.\,\phi(\alpha)^{c_1+1}\phi'(\alpha)\right.$$

$$\left.-\frac{d\beta}{dx}\,\epsilon^{x_1\phi(\beta)-c_2'\phi(\beta)}\,.\,\phi(\beta)^{c_1+1}\phi'(\beta)\right\}.$$

Solution by the PROPOSER.

Let α, β, x_1 be any functions of x, and $u=\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,\psi(z)\,dz+c$ ......(2);

then, differentiating (2) with respect to (x), (TODHUNTER's *Int. Calc.*, p. 198)

$$\frac{du}{dx}=\int_\beta^\alpha\frac{d}{dx}\left[\epsilon^{x_1\phi(z)}\,.\,\psi(z)\right]dz+\frac{d\alpha}{dx}\,\epsilon^{x_1\phi(\alpha)}\,.\,\psi(\alpha)-\frac{d\beta}{dx}\,\epsilon^{x_1\phi(\beta)}\,.\,\psi(\beta)$$

or
$$\frac{du}{dx}=\frac{dx_1}{dx}\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)\,\psi(z)\,dz+\mathrm{N}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$
where
$$\mathrm{N}=\frac{d\alpha}{dx}\,\epsilon^{x_1\phi(\alpha)}\,\psi(\alpha)-\frac{d\beta}{dx}\,\epsilon^{x_1\phi(\beta)}\,.\,\psi(\beta)\ \ldots\ldots\ldots\ldots\ldots(4).$$

Differentiating (3) with respect to x, we have
$$\frac{d^2u}{dx^2}=\frac{d\mathrm{N}}{dx}+\mathrm{M}+\frac{d^2x_1}{dx^2}\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)\,\psi(z)\,dz+\left(\frac{dx_1}{dx}\right)^2\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)^2\psi(z)\,dz$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ (5)$$
where
$$\mathrm{M}=\frac{d\alpha}{dx}\,.\,\frac{d}{dx}\left[\epsilon^{x_1\phi(\alpha)}\,.\,\psi(\alpha)\right]-\frac{d\beta}{dx}\,.\,\frac{d}{dx}\left[\epsilon^{x_1\phi(\beta)}\,.\,\psi(\beta)\right]$$
$$=\frac{dx_1}{dx}\left\{\frac{d\alpha}{dx}\,\epsilon^{x_1\phi(\alpha)}\,\phi(\alpha)\,\psi(\alpha)-\frac{d\beta}{dx}\,\epsilon^{x_1\phi(\beta)}\,.\,\phi(\beta)\,\psi(\beta)\right\}\ldots\ldots(6).$$

From (3) and (5) we obtain
$$\left.\begin{aligned}\frac{d^2u}{dx^2}&=\frac{d\mathrm{N}}{dx}+\mathrm{M}+\frac{d^2x_1}{dx^2}\left(\frac{du}{dx}\Big/\frac{dx_1}{dx}-\mathrm{N}\right)+\left(\frac{dx_1}{dx}\right)^2\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)^2\psi(z)\,dz\\&=\frac{d\mathrm{N}}{dx}+\mathrm{M}+\frac{d^2x_1}{dx^2}\Big/\frac{dx_1}{dx}\,.\,\frac{du}{dx}-\mathrm{N}\,\frac{d^2x_1}{dx^2}+\left(\frac{dx_1}{dx}\right)^2\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)^2\psi(z)\,dz\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots(7).$$

Integrating the part affected by the integral sign in (7) by parts in the usual way, we have
$$\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\phi(z)^2\psi(z)\,dz=\int_\beta^\alpha\frac{\phi(z)^2\,\psi(z)}{x\phi'(z)}\,\frac{d}{dz}\left[\epsilon^{x_1\phi(z)}\right]dz$$
$$=\mathrm{L}-\int_\beta^\alpha\frac{\epsilon^{x_1\phi(z)}}{x_1}\left\{2\phi(z)\,\psi(z)+\frac{\phi(z)^2\,\psi'(z)}{\phi'(z)}-\frac{\phi(z)^2\,\psi(z)\,\phi''(z)}{\phi''(z)}\right\}dz$$
$$=\mathrm{L}-\frac{2}{x_1}\left(\frac{du}{dx}\Big/\frac{dx_1}{dx}-\mathrm{N}\right)-\frac{1}{x_1}\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\frac{\phi(z)^2}{\phi'(z)}\left\{\psi'(z)-\frac{\psi(z)\,\phi''(z)}{\phi'(z)}\right\}dz$$
$$=\mathrm{L}-\frac{2}{x_1\,dx_1/dx}\,.\,\frac{du}{dx}+\frac{2\mathrm{N}}{x_1}-\frac{1}{x_1}\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\frac{\phi(z)^2}{\phi'(z)}\left\{\psi'(z)-\frac{\psi(z)\,\phi''(z)}{\phi'(z)}\right\}dz$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(8);$$
where
$$\mathrm{L}=\frac{1}{x_1}\left\{\frac{\epsilon^{x_1\phi(\alpha)}\,.\,\phi(\alpha)^2\psi(\alpha)}{\phi'(\alpha)}-\frac{\epsilon^{x_1\phi(\beta)}\,.\,\phi(\beta)^2\,\psi(\beta)}{\phi'(\beta)}\right\}\ldots\ldots\ldots(9).$$

Substitute the value of (8) in equation (7); then
$$\left.\begin{aligned}&\frac{d^2u}{dx^2}+\left(2\frac{dx_1}{x_1dx}-\frac{dx_1}{dx^2}\Big/\frac{dx_1}{dx}\right)\frac{du}{dx}\\&=\left(\frac{2\,dx_1}{x_1\,dx}-\frac{d^2x_1}{dx^2}\Big/\frac{dx_1}{dx}\right)\mathrm{N}\,\frac{dx_1}{dx}+\frac{d\mathrm{N}}{dx}+\mathrm{M}+\left(\frac{dx_1}{dx}\right)^2\mathrm{L}\\&-\frac{1}{x_1}\left(\frac{dx_1}{dx}\right)^2\int_\beta^\alpha \epsilon^{x_1\phi(z)}\,.\,\frac{\phi^2(z)}{\phi'(z)}\left\{\psi'(z)-\frac{\psi(z)\,\phi''(z)}{\phi'(z)}\right\}dz\end{aligned}\right\}\ldots\ldots(10).$$

In equation (10) take
$$\frac{\phi(z)^2}{\phi'(z)}\left\{\psi'(z)-\frac{\psi(t)\,\phi''(t)}{\phi'(z)}\right\}=c_1\,\phi(z)\,\psi(z)+c_2\,\psi(z)\ldots\ldots\ldots(11),$$

then we have $\qquad \psi(z) = c_3 \epsilon^{-c_2'\phi(z)} \, . \, \phi(z)^{c_1} \phi'(z)$ ....(12).

Observing the conditional equation (11), equation (10) becomes

$$\left.\begin{array}{l} \dfrac{d^2u}{dx^2} + \left(\dfrac{2+c_1}{x_1}\dfrac{dx_1}{dx} - \dfrac{d^2x_1}{dx^2}\Big/\dfrac{dx_1}{dx}\right)\dfrac{du}{dx} + \dfrac{c_1}{x_1}\left(\dfrac{dx_1}{dx}\right)^2 u \\ = \left(\dfrac{2+c_1}{x_1}\cdot\dfrac{dx_1}{dx} - \dfrac{d^2x_1}{dx^2}\Big/\dfrac{dx_1}{dx}\right) \mathrm{N}\dfrac{dx_1}{dx} + \dfrac{d\mathrm{N}}{dx} + \mathrm{M} + \dfrac{c_2 c}{x_1}\left(\dfrac{dx_1}{dx}\right)^2 + \left(\dfrac{dx_1}{dx}\right)^2 \mathrm{L} \end{array}\right\} \quad \text{..................(13).}$$

The integral of equation (13) leads therefore, observing the equation (12), to the results in the Question, the constants M, N being as there given, and

$$\mathrm{L} = \frac{c_3}{x_1}\left\{\epsilon^{x_1\phi(\alpha) - c_2/\phi(\alpha)} \, . \, \phi(\alpha)^{c_1+2} - \epsilon^{x_1\phi(\beta) - c_2/\phi(\beta)} \, . \, \phi(\beta)^{c_1+2}\right\}.$$

[By assuming the particular values, $x_1 = -a^n x^n$, $c = 0$, $c_2 = 1$, $nc_3 = -1$, $c_1 n = -n-1$, $\phi(z) = z^{-n}$, $\alpha = \infty$, $\beta = a^{\frac{1}{2}} x^{\frac{1}{2}}$, we obtain herefrom a solution of Question 5400; see Vol. XXVIII, p. 76].

8020. (Asparagus.)—A conic circumscribes a given triangle ABC and one focus lies on BC; prove that the envelop of the corresponding directrix is a conic with respect to which A is the pole of BC; and, if A be a right angle, the envelop is the parabola whose focus is A and directrix BC. [If $(0, 0)$, (a, b), $(a, -c)$ are the coordinates of A, B, C, the equation of the envelop will be

$$4bc(bc-a^2)x^2 + 4a(b+c)(bc-a^2)xy + a^2[4a^2+(b-c)^2]y^2 + a^2(b+c)^2(2ax-a^2) = 0.]$$

Solution by R. Lachlan, M.A.

1. Let $px+qy+rz = 0$, be the equation in areal coordinates of the directrix of a conic which passes through the angular points of the triangle of reference ABC. If S be the corresponding focus, and e the eccentricity, we have at once $\mathrm{SA} = ep$, $\mathrm{SB} = eq$, $\mathrm{SC} = er$.

2. If S lie on a circle, we have $l\,.\,\mathrm{SA}^2 + m\,.\,\mathrm{SB}^2 + n\,.\,\mathrm{SC}^2 = k$.

And, if $l+m+n = 0$, then S lies on a straight line; thus we shall have

$$lp^2 + mq^2 + nr^2 = k/e^2.$$

3. Again, S, A, B, C being four points in a plane, we have

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & \mathrm{SA}^2, & \mathrm{SB}^2, & \mathrm{SC}^2 \\ 1, & \mathrm{SA}^2, & 0, & \mathrm{AB}^2, & \mathrm{AC}^2 \\ 1, & \mathrm{SB}^2, & \mathrm{AB}^2, & 0, & \mathrm{BC}^2 \\ 1, & \mathrm{SC}^2, & \mathrm{AC}^2, & \mathrm{BC}^2, & 0 \end{vmatrix} = 0; \quad \therefore \quad \begin{vmatrix} 0, & 1/e^2, & 1, & 1, & 1 \\ 1/e^2, & 0, & p^2, & q^2, & r^2 \\ 1, & p^2, & 0, & c^2, & b^2 \\ 1, & q^2, & c^2, & 0, & a^2 \\ 1, & r^2, & b^2, & a^2, & 0 \end{vmatrix} = 0.$$

4. If then the locus of S be $l\,.\,\mathrm{SA}^2+m\,.\,\mathrm{SB}^2+n\,.\,\mathrm{SC}^2=k$,
the tangential equation of the envelop of the corresponding directrix is

$$\begin{vmatrix} 0, & lp^2+mq^2+nr^2, & k, & k, & k \\ lp^2+mq^2+nr^2, & 0, & p^2, & q^2, & r^2 \\ k, & p^2, & 0, & c^2, & b^2 \\ k, & q^2, & c^2, & 0, & a^2 \\ k, & r^2, & b^2, & a^2, & 0 \end{vmatrix}=0;$$

which may also be written

$$\begin{vmatrix} u, & 0, & k-c^2m-b^2n, & k-c^2l-a^2n, & k-b^2l-a^2m \\ 0, & 0, & p^2, & q^2, & r^2 \\ k-c^2m-b^2n, & p^2, & 0, & c^2, & b^2 \\ k-c^2l-a^2n, & q^2, & c^2, & 0, & a^2 \\ k-b^2l-a^2m, & r^2, & b^2, & a^2, & 0 \end{vmatrix}=0,$$

where $\qquad u=2\,(a^2mn+b^2nl+c^2lm)-2k\,(l+m+n).$

Thus, if the locus of a focus of a system of conics circumscribing a triangle be a circle or a straight line, the envelop of the corresponding directrix is a curve of the fourth class.

5. If the locus of the focus be the straight line BC, we have

$$a\,.\,\mathrm{SA}^2-b\cos\mathrm{C}\,.\,\mathrm{SB}^2-c\cos\mathrm{B}\,.\,\mathrm{SC}^2=abc\cos\mathrm{A},$$

and the envelop of the directrix is

$$\begin{vmatrix} -8\Delta^2, & 0, & \frac{8\Delta^2}{a}, & 0, & 0 \\ 0, & 0, & p^2, & q^2, & r^2 \\ \frac{8\Delta^2}{a}, & p^2, & 0, & c^2, & b^2 \\ 0, & q^2, & c^2, & 0, & a^2 \\ 0, & r^2, & b^2, & a^2, & 0 \end{vmatrix}=0;$$

which reduces to $\qquad \left\{-p^2a^2+q^2b^2+r^2c^2\right\}^2-4b^2c^2q^2r^2\cos^2\mathrm{A}=0.$

Thus, if the focus of a conic circumscribing the triangle ABC lie on BC, the envelop of the corresponding directrix consists of the two curves
$-p^2a^2+q^2b^2+r^2c^2+2bc\cos\mathrm{A}\,qr=0,\ -p^2a^2+q^2b^2+r^2c^2-2bc\cos\mathrm{A}\,qr=0$ (1, 2).

If D, E, F be the mid-points of ABC, (1) is clearly touched by DE and DF, and (2) by EF and the line at ∞. Thus (2) is a parabola.

Again, it is clear that A is the pole of BC with respect to (1) and (2).

If A be a right angle, (1) and (2) coincide, and the envelop is clearly the parabola whose focus is A and directrix BC.

6. If the focus lies on the circum-circle of ABC, we shall have

$$l\,.\,\mathrm{SA}^2+m\,.\,\mathrm{SB}^2+n\,.\,\mathrm{SC}^2=k,$$

where $\qquad mc^2+nb^2=k, \qquad lc^2+na^2=k, \qquad lb^2+ma^2=k;$

hence the envelope of the directrix is

$$\begin{vmatrix} 0, & p^2, & q^2, & r^2 \\ p^2, & 0, & c^2, & b^2 \\ q^2, & c^2, & 0, & a^2 \\ r^2, & b^2, & a^2, & 0 \end{vmatrix}=0; \qquad \text{or} \qquad pa\pm qb\pm rc=0.$$

Thus, if the focus of a system of circum-conics lie on the circum-circle, the corresponding directrix passes through one of the centres of the circles which touch the sides of the triangle.

7. More generally, if the locus of S be given by an equation of the form
$$u_n + u_{n-1} + \ldots + u_0 = 0,$$
where u_n is a homogeneous function of SA, SB, SC of the nth degree, then, substituting SA $= ep$, &c., we shall have
$$e^n \cdot f_n(pqr) + e^{n-1} f_{n-1}(pqr) + \ldots = 0.$$
And if e be eliminated from this and the equation in § 3, we obtain the tangential equation of the envelop of the directrix corresponding to S.

8. If the locus of S be given by an equation of the form
$$f(\mathrm{SA}, \mathrm{SB}, \mathrm{SC}) = 0,$$
where f is homogeneous of the nth degree, then clearly the envelop of the directrix is $f(p, q, r) = 0$, a curve of the nth class, and conversely.

7949. (R. KNOWLES, B.A.)—Prove that the sum of the series $\frac{1}{2}x - \frac{1}{5}x^2 + \frac{1}{8}x^3 - \ldots$ *ad. inf.*
$$= 3^{-1} x^{\frac{1}{3}} \log \frac{(1 - x^{\frac{1}{3}} + x^{\frac{2}{3}})^{\frac{1}{2}}}{1 + x^{\frac{1}{3}}} + 3^{-\frac{1}{2}} x^{\frac{1}{3}} \left\{ \tan^{-1} \frac{2x^{\frac{1}{3}} - 1}{3^{\frac{1}{2}}} + \cot^{-1} 3^{\frac{1}{2}} \right\}.$$

Solution by Rev. T. C. SIMMONS, M.A.; J. O'REGAN; *and others.*

Putting $x = y^3$, and dividing by y, the left-hand side becomes
$$\tfrac{1}{2}y^2 - \tfrac{1}{5}y^5 + \tfrac{1}{8}y^8 - \ldots \equiv \mathrm{S}.$$
$$\frac{d\mathrm{S}}{dy} = y - y^4 + y^7 - \ldots = \frac{y}{1 + y^3}, \quad \mathrm{S} = \int \frac{y}{1 + y^3} = \tfrac{1}{3} \int \frac{y + 1}{y^2 - y + 1}\, dy - \tfrac{1}{3} \int \frac{dy}{y + 1}$$
$$= \tfrac{1}{3} \log \frac{(y^2 - y + 1)^{\frac{1}{2}}}{y + 1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2y - 1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cot^{-1} \sqrt{3};$$
the constant being added in order to make S and y vanish together.

Hence the stated sum of the original series follows.

8668. (ALPHA.)—The ellipse whose eccentricity is $\frac{1}{3}\sqrt{2}$ is referred to the triangle formed by joining a focus to the extremities of the latus rectum through the other focus: prove that its equation is
$$\gamma^2 + 9(\beta\gamma + \gamma\alpha + \alpha\beta) = 0.$$

Solution by A. GORDON; Rev. T. GALLIERS, M.A.; *and others.*

If $x^2/a^2 + y^2/b^2 - 1 = 0$ be the ellipse referred to its axes, we have
$$-x + y\,2\sqrt{2} - 3p = \alpha, \quad -x - y\,2\sqrt{2} - 3p = \beta, \quad x - 3p = \gamma \ldots\ldots (1, 2, 3),$$
and $\cos\alpha = -\frac{1}{3}$, $p = a/3\sqrt{2}$, $x^2 + 2y^2 = 18p^2$ .... (4).

Eliminating x, y, p between (1), (2), (3), and (4), we obtain the result.

9449. (Professor SYLVESTER, F.R.S.)—If there exist any perfect number divisible by a prime number p of the form 2^n+1, show that it must be divisible by another prime number of the form $px \pm 1$.

Solution by W. S. FOSTER.

Let the number $N = p^a \,.\, q^b \,.\, r^c \ldots$; then, since N is a perfect number, we must have one of the factors (say, q^b) such that $q^{b+1}-1$ is divisible by p, therefore $q^{b+1} = M(p)+1$; and, since p and q are prime numbers, $q^{p-1} = M(p)+1$, therefore $b+1$ is a divisor of $2^n = 2^z$ suppose. Let $q = xp \pm h$, then $h^{2^z}-1 = M(p)$; hence h must be some power of the remainder after dividing $(2^n)^{2^n-z}$ by 2^n+1; therefore h must equal 1, and $q = xp \pm 1$, which is a prime divisor of N.

9468. (R. W. D. CHRISTIE, M.A.)—Show that the tenth perfect number is $P_{10} = 2^{40}(2^{41}-1) = 2,417,851,639,228,158,837,784,576$.

Solution by Professor IGNACIO BEYENS.

Le dixième nombre parfait est donné par M. CARVALLO dans l'ouvrage *Théorie des Nombres parfaits*.

[Every divisor of 2^p-1 is of form $2px+1$ when p is a prime; but $2^{41}-1$ is indivisible by $82x+1$; hence $2^{40}(2^{41}-1) = $ &c. is a perfect number.]

3419. (ARTEMAS MARTIN.)—The point A_1 is taken at random in the side BC of a triangle ABC, B_1 in CA, and C_1 in AB; the point A_2 is taken at random in the side B_1C_1 of the triangle $A_1B_1C_1$, B_2 in C_1A_1, and C_2 in A_1B_1, and so on; find the average area of the triangle $A_nB_nC_n$.

Solution by D. BIDDLE.

It is difficult to understand why this question has remained unanswered so long; but the reason may be any one of three:—(1) its extreme simplicity, (2) the fear of some concealed pitfall, or (3) a mere disinclination to consider the matter. Unless (2) be well grounded, there can be no doubt that the correct answer is $(\frac{1}{4})^n$ABC. For, let $\Delta_1, \Delta_2, \ldots \Delta_n$ represent the successive triangles drawn at random upon the given ABC, in the way described. Then, the average area of Δ_n will be $\frac{1}{4}\Delta_{n-1}$; of Δ_{n-1}, $\frac{1}{4}\Delta_{n-2}$; and so on, until by retrogression we arrive at Δ_1, which on the average is $\frac{1}{4}$ABC. The fact, in regard to each pair taken separately, is well known. But, if Δ_2 be on the average $\frac{1}{4}$ of Δ_1, which on the average is $\frac{1}{4}$ABC, it seems impossible to escape from the conclusion that on the average $\Delta_2 = (\frac{1}{4})^2$ ABC. In being $\frac{1}{4}$, on the average, of *any* Δ_1, on which it may be drawn, Δ_2 is on the average $\frac{1}{4}$ of $\frac{1}{4}$ABC. It is a case of multiple integration in which, as each variable is eliminated, the additional factor $\frac{1}{4}$ is yielded to the result.

9462. (The EDITOR.)—If the radius of the in-circle of an isosceles triangle is one-n^{th} of the radius of the ex-circle to the base; prove that the ratio of the base to each of the equal sides is $2(n-1) : n+1$.

Solution by Professors EMMERICH, Ph.D.; IGNACIO BEYENS; *and others.*

Draw perpendiculars TD, $T_a D_a$ from the centres T, T_a of the in-circle and the ex-circle to the base on AB. From similar triangles, we have $AD : AD_a = 1 : n$. But $AD = \frac{1}{2}(2b-a)$, $AD_a = \frac{1}{2}(2b+a)$, a denoting the base, and b the other sides; therefore $2b+a : 2b-a = n : 1$; hence

$$2a : 4b = n-1 : n+1, \quad \text{and} \quad a : b = 2(n-1) : n+1.$$

9140. (Rev. T. C. SIMMONS, M.A.)—Prove *geometrically* that the perpendicular from the Lemoine-point of an harmonic polygon on the Lemoine-line is the harmonic mean of the perpendiculars drawn on the same line from the vertices of the polygon. [A proof by trigonometrical series is given in *Lond. Math. Soc. Proceedings*, Vol. XVIII., p. 293.]

Solution by R. F. DAVIS, M.A.

If a polygon ABC ... L (n sides) inscribed in an ellipse is such that each side subtends the same angle ($2\pi/n$) at the focus S; then, projecting orthogonally, we get a harmonic polygon $abc \dots l$ (n sides) inscribed in a circle whose Lemoine-point s is the projection of S, and whose Lemoine-line yy' is the projection of the S-directrix YY′. This property is the basis of Mr. SIMMONS' theory of harmonic polygons, as set forth in the above paper.

The properties of the polygon ABC ... L may be derived in turn by reciprocating with respect to any point S a regular polygon of n sides circumscribing a circle. Since (by a well-known theorem) the sum of the perpendiculars of S upon the sides of the latter polygon $= n$ (radius), we have $\Sigma(k^2/SA) = n(k^2/SD)$, where SD is the semi latus-rectum of the ellipse. Hence, if SX, AA′, BB′ ... be perpendiculars upon YY′, SX is the harmonic mean of AA′, BB′ ...; and this relation is unaltered by projection.

8968. (W. J. C. SHARP, M.A.)—If (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3) be any three points, and λ, μ, ν the areal coordinates of any point in their plane referred to the triangle of which they are vertices; show that the equation to the section of any surface $U = 0$ by the plane will be obtained by substituting for x, y, z, w from the equations

$$(\lambda+\mu+\nu)\, x = \lambda x_1 + \mu x_2 + \nu x_3, \quad (\lambda+\mu+\nu)\, y = \lambda y_1 + \mu y_2 + \nu y_3,$$

$$(\lambda+\mu+\nu)\, z = \lambda z_1 + \mu z_2 + \nu z_3, \quad (\lambda+\mu+\nu)\, w = \lambda w_1 + \mu w_2 + \nu w_3.$$

Solution by D. EDWARDES.

Let Δ be the area of the triangle formed by the three points. Then any point in the plane of the triangle may be expressed

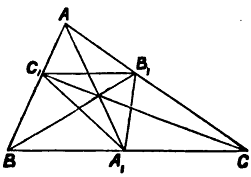
$$\frac{lx_1+mx_2+nx_3}{l+m+n},\ \frac{ly_1+my_2+ny_3}{l+m+n},\ \&c.$$

Also $\lambda = \dfrac{l}{l+m+n}\Delta$, &c., therefore $\Delta x = \lambda x_1+\mu x_2+\nu x_3$, $\Delta y =$ &c. &c.; and $\Delta = \lambda+\mu+\nu$, therefore, &c.

[That the point $\left(\dfrac{lx_1+mx_2+nx_3}{l+m+n},\ \dfrac{ly_1+my_2+ny_3}{l+m+n}\ldots\right)$ is a point in the plane follows at once, because if this point be $(x, y \ldots)$ it satisfies the equation $\begin{vmatrix} x, & x_1, & x_2, & x_3 \\ y, & y_1, & y_2, & y_3 \\ z, & z_1, & z_2, & z_3 \\ w, & w_1, & w_2, & w_3 \end{vmatrix} = 0$, which is the equation to the plane of the triangle.]

9350. (Professor DE WACHTER.)—A point being taken within a triangle, prove that the chance that its distances from the sides (a), (b), (c), may form any possible triangle will be $2abc/\{(b+c)(c+a)(a+b)\}$.

Solution by Professor IGNACIO BEYENS.



Soient A_1, B_1, C_1 les pieds des bissectrices du triangle ABC; il est aisé à démontrer que la droite B_1C_1 est le lieu géométrique des points tels que leur distance au côté BC est égale à la somme des distances aux autres deux côtés AB, AC, et que par suite pour tout autre point situé dans l'intérieur du triangle AB_1C_1 la distance à BC est plus grande que la somme des autres distances à AB, AC, et que pour un point du quadrilatère BCB_1C_1 la distance à BC est plus petite que la somme des distances à AB, AC. La même chose arrivera aux droites B_1A_1, A_1C_1, et par suite tout point intérieur à $A_1B_1C_1$ aura la propriété que l'une quelconque de ses distances aux côtés AB, AC, BC, sera plus petite que la somme des deux autres, et par conséquence la probabilité demandée sera $\Delta A_1B_1C_1 : \Delta ABC = 2abc : (b+c)(c+a)(a+b)$.

8344. (R. KNOWLES, B.A.)—AD, BE, CF are drawn from the angular points of a triangle ABC, so that the angles BAD, EBC, ACF are each equal to the Brocard-angle of the triangle; show that their equations are

$$bcy-a^2z=0,\quad b^2x-acz=0,\quad abx-c^2y=0.$$

Solution by GEORGE GOLDTHORPE STORR, M.A.

If ω be the Brocard angle of the triangle, we have

$$\cot\omega = \cot A + \cot B + \cot C.$$

Now the equation to AD is

$$\frac{y}{z} = \frac{\sin(A-\omega)}{\sin\omega} = \sin A\cot\omega - \cos A = \frac{\sin^2 A}{\sin B \sin C} = \frac{a^2}{bc}, \text{ or } bcy - a^2z = 0.$$

Similarly for the equations to BE and CF.

9376. (A. E. THOMAS.)—Solve the equations

$$x^4 + 3y^2z^2 = a^4 + 2x(y^3+z^3) \quad\text{.........................(1)},$$

$$y^4 + 3z^2x^2 = b^4 + 2y(z^3+x^3), \quad z^4 + 3x^2y^2 = c^4 + 2z(x^3+y^3) \quad\text{......(2, 3)}.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

Adding, $x^2+y^2+z^2-xy-yz-zx \equiv (x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$$= (a^4+b^4+c^4)^{\frac{1}{2}} \quad\text{......................(4)},$$

(2) $\times\,\omega$, (3) $\times\,\omega^2$ give $(x+y+z)(x+\omega^2 y+\omega z) = (a^4+\omega b^4+\omega^2 c^4)^{\frac{1}{2}}$ ......(5),

$$(x+y+z)(x+\omega y+\omega^2 z) = (a^4+\omega^2 b^4+\omega c^4)^{\frac{1}{2}} \quad\text{...............(6)}.$$

$$\frac{(5)\times(6)}{(4)} \text{ gives } x+y+z = \frac{(a^4+\omega b^4+\omega^2 c^4)^{\frac{1}{2}}\,(a^4+\omega^2 b^4+\omega c^4)^{\frac{1}{2}}}{(a^4+b^4+c^4)^{\frac{1}{2}}},$$

with similar expressions for $x+\omega y+\omega^2 z$, &c.; adding, we obtain x in a form that can easily be rationalized, and then the values of y, z can be written down.

9430. (Professor WOLSTENHOLME, M.A., Sc.D.)—In a tetrahedron OABC, the plane angles of the triangular faces are denoted by α, β, or γ; all angles opposite to OA or BC being α, those opposite OB or CA are β, and those opposite OC or AB are γ; the angles at O have the suffix 1, those at B, C, D the suffixes 2, 3, 4 respectively; prove that, if $\alpha_1+\beta_1+\gamma_1 = \alpha_2+\beta_2+\gamma_2 = \pi$, then

$$\gamma_1+\alpha_1-\beta_1 = \gamma_4+\alpha_4-\beta_4; \quad \alpha_1+\beta_1-\gamma_1 = \alpha_3+\beta_3-\gamma_3$$

$$\gamma_2+\alpha_2-\beta_2 = \gamma_4+\alpha_3-\beta_3; \quad \alpha_2+\beta_2-\gamma_2 = \alpha_4+\beta_4-\gamma_4.$$

Solution by Professor SWAMINATHA AIYAR, B.A.

C_1, C_2, C_3 are any three points not in the same straight line. O is the middle point of C_1, C_2, A of C_1, C_3, and B any point in the plane equidistant from C_2 and C_3. Now the tetrahedron of which the faces are the triangles OAB, OAC_1, OBC_2, ABC_3 is of the sort described in the question, and, naming the angles as directed, we see at once from the figure, since OA is parallel to C_2C_3, that

$$\alpha_1+\gamma_1-\beta_1 = \alpha_4+\gamma_4-\beta_4; \quad \alpha_2+\beta_2-\gamma_2 = \alpha_4+\beta_4-\gamma_4.$$

A similar proof is easily seen to hold for the other part.

[Professor WOLSTENHOLME remarks that he had not attempted a deductive proof of this theorem; the lengths of the edges of the tetrahedron, by means of which he noticed the property, are

$$\begin{array}{ll} DA = 7{\cdot}069273, & BC = 7{\cdot}13973, \\ DB = 7{\cdot}375329, & CA = 6{\cdot}4544, \\ DC = 7{\cdot}315125, & AB = 8{\cdot}13924; \end{array}$$

the half angles at D are $29°4'40''{\cdot}45$, the half angles at A are $28°35'42''{\cdot}85$,

$$\begin{array}{rlrl} & 21°38'39''{\cdot}03 & & 28°45'\ 4''{\cdot}73 \\ & \underline{34°16'40''{\cdot}52} & & \underline{32°39'12''{\cdot}42} \\ \sigma_1 = & 90°\ \ 0'\ \ 0'' & \sigma_2 = & 90°\ \ 0'\ \ 0'' \end{array}$$

those at B are $26°\,58'\,14''{\cdot}75$, those at C are $30°\,42'\ \ 8''{\cdot}55$;

$$\begin{array}{rlrl} & 24°\,43'\,23''{\cdot}96 & & 30°\,40'\,19''{\cdot}78 \\ & \underline{30°\,14'\,59''{\cdot}77} & & \underline{36°\,40'\,53''{\cdot}19} \\ \sigma_3 = & 81°\,56'\,38''{\cdot}48 & \sigma_4 = & 98°\ \ 3'\ \ 21''{\cdot}52 \end{array}$$

the dihedral angle opposite DA is $66°\ \ 1'\ \ 3''{\cdot}76$ }
,, ,, BC is $65°\,31'\,50''{\cdot}4$ },

opposite DB $= 65°\,58'\,29''{\cdot}34$ }, opposite DC is $79°\,42'\,30''{\cdot}7$ }
,, CA $= 59°\,11'\,49''{\cdot}52$ }, ,, AB is $85°\,46'\,22''{\cdot}96$ },

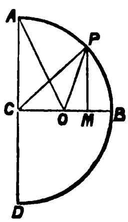
$$\left.\begin{array}{l}\gamma_1+\alpha_1-\beta_1 = 73°\,25'\,23''{\cdot}88 \\ \gamma_4+\alpha_4-\beta_4 = 73°\,25'\,23''{\cdot}92\end{array}\right\}, \quad \left.\begin{array}{l}\gamma_2+\alpha_2-\beta_2 = 64°\,59'\,41''{\cdot}08 \\ \gamma_3+\alpha_3-\beta_3 = 64°\,59'\,41''{\cdot}12\end{array}\right\},$$

$$\left.\begin{array}{l}\alpha_1+\beta_1-\gamma_1 = 42°\,53'\,17''{\cdot}92 \\ \alpha_3+\beta_3-\gamma_3 = 42°\,53'\,17''{\cdot}96\end{array}\right\}, \quad \left.\begin{array}{l}\alpha_2+\beta_2-\gamma_2 = 49°\,23'\,10''{\cdot}32 \\ \alpha_4+\beta_4-\gamma_4 = 49°\,23'\,10''{\cdot}28\end{array}\right\},$$

the difference in each case being $''{\cdot}04$; which is as near as can be expected with 7 figure logarithms.]

9006. (H. L. ORCHARD, B.Sc., M.A.)—Inside a hemisphere (of radius ρ) a luminous point is placed, in the radius which is perpendicular to the base, at a distance from the base $= \frac{1}{3}\rho\sqrt{3}$; show that the illumination of the surface (excluding the base) is $= 3\pi C$.

Solution by Rev. T. GALLIERS, M.A.



Let O be the luminous point; ABD the vertical section of the hemisphere (its centre being at C) through CO; $\angle CPO = \phi$; $\angle POB = \theta$; radius of hemisphere $= a$; $CO = c$, $a = c\sqrt{3}$; also let $OP = r$.

Then the illumination of a band generated by the revolution of the elementary arc at P about CB

$$= 2\pi C\,(r\sin\theta\,.\cos\phi\,ds)/r^2 = dI \text{ (say)}.$$

Now $\cos\phi = \dfrac{a^2+r^2-c^2}{2ar}$, also $a^2 = r^2+c^2+2cr\cos\theta$;

therefore $$(r+c\cos\theta)\,dr = cr\sin\theta\,.d\theta,$$

or $$\frac{r\,d\theta}{dr} = \frac{r + c\cos\theta}{c\sin\theta}, \text{ therefore } \frac{ds}{dr} = \frac{a}{c\sin\theta};$$

thus $$dI = \frac{\pi C}{c}\left\{\frac{a^2-c^2}{r^2}+1\right\}dr;$$

therefore illumination of hemi-spherical surface

$$= \frac{\pi C}{c}\int_{a-c}^{(a^2+c^2)^{\frac{1}{2}}}\left\{\frac{a^2-c^2}{r^2}+1\right\}dr = 3\pi C, \text{ the result given.}$$

9433. (G. Heppel, M.A.)—If, within a triangle ABC, O be a point where the sides subtend equal angles; then, putting OA $= p$, OB $= q$, OC $= r$, show that the equation to the ellipse with focus O, touching the sides in D, E, F, is in (1) rectangular coordinates, with O as origin and OA as axis of y, and (2) trilinear coordinates, ABC triangle of reference,

$$(x^2+y^2)^{\frac{1}{2}} = \tfrac{1}{2}(pq+qr+rp)^{-1}\left[(pr+pq-2qr)\,y - p\,(q-r)\,x\sqrt{3} + 3pqr\right] \ldots(1),$$

$$a^2p^2\alpha^2 + b^2q^2\beta^2 + c^2r^2\gamma^2 - 2bc\,qr\,\beta\gamma - 2ca\,rp\,\gamma\alpha - 2ab\,pq\,\alpha\beta = 0 \ldots\ldots(2).$$

Solution by W. S. Foster.

Since O is the focus of the ellipse touching the sides of the triangle, the angle AOE = AOF, and AOB = AOC, therefore BOF = COE, therefore BOD = COD, therefore OD bisects the angle BOC, therefore AOD is a straight line. Let $L\alpha^2 + M\beta^2 + N\gamma^2 - 2LM\alpha\beta - 2LN\alpha\gamma - 2MN\beta\gamma = 0$ be the equation to the ellipse; then the line AD is $M\beta - N\gamma = 0$, and since this passes through O, whose coordinates are given by the equations

$$ap\;\alpha = bq\,.\,\beta = cr\,.\,\gamma, \quad \therefore \frac{M}{N} = \frac{bq}{cr}, \quad \therefore \frac{M}{bq} = \frac{N}{cr} = \text{also } \frac{L}{ap}.$$

Substituting these values of L, M, N in the equation to the ellipse, we have the equation given in the question for trilinear coordinates.

Let $k/\rho = 1 + e\cos\theta$ be the equation to the ellipse, referred to the major axis as initial line, and let this make an angle θ_1 with OA; then, since AE touches the ellipse at E,

$$\frac{k}{p} = e\cos\theta_1 + \cos \text{AOE} = e\cos\theta_1 + \tfrac{1}{2},$$

and $$\frac{k}{q} = e\cos\left(\frac{2\pi}{3}+\theta_1\right) + \tfrac{1}{2} = e\left(-\tfrac{1}{2}\cos\theta_1 - \frac{\sqrt{3}}{2}\sin\theta_1\right) + \tfrac{1}{2},$$

$$\frac{k}{r} = e\cos\left(\frac{4\pi}{3}+\theta_1\right) + \tfrac{1}{2} = e\left(-\tfrac{1}{2}\cos\theta_1 + \frac{\sqrt{3}}{2}\sin\theta_1\right) + \tfrac{1}{2},$$

therefore

$$k = \frac{3pqr}{2\,(pq+qr+rp)}, \quad e\cos\theta_1 = \frac{2qr-pr-pq}{2\,(pq+qr+pr)}, \quad e\sin\theta_1 = \frac{\sqrt{3}\,p\,(q-r)}{2\,(pq+qr+pr)};$$

therefore the equation to the ellipse referred to a line perpendicular to OA is $$k/\rho = 1 + e\sin(\theta_1+\theta),$$

$$\therefore\; (x^2+y^2)^{\frac{1}{2}} = \tfrac{1}{2}(pq+qr+pr)^{-1}\left\{(pr+pq-2qr)\,y - \sqrt{3}\,p\,(q-r)\,x + 3pqr\right\}.$$

8461. (F. R. J. HERVEY.) — Find in how many ways n lines of verse can be rhymed, supposing that (1) no line be left unrhymed, and (2) the restriction as to unrhymed lines be removed; and show that, in the case of the *sonnet*, the respective numbers of ways are 24011157 and 190899322.

Solution by the PROPOSER.

Let the numbers of ways be denoted, (1) by ϕn, (2) by fn; and first consider fn. Let line-endings which rhyme together be considered the same, and denoted by the same letter. Let the first ending be denoted by a, the next *distinct* ending by b, the third by c, and so on. Two lines give two cases, aa and ab. Three lines give five, for aa may be followed by a or b, and ab by a, b, or c. Generally, an arrangement of this sort containing only the first r letters of the alphabet may be followed by any one of the first $r+1$ letters; and the complete sets of arrangements for successive values of n may be exhibited in a table such as the following. Each letter gives rise to a *group* in the next line; a group of r letters, wherever found, gives invariably $r-1$ groups of r followed by a group of $r+1$; and the number of letters in the n^{th} line is fn. I shall show that $fn = \Delta^n f1$; this is contained in the two following propositions.

<table>
<tr><td>a</td><td colspan="5"></td></tr>
<tr><td>a</td><td colspan="5">b</td></tr>
<tr><td>a   b</td><td>a</td><td colspan="2">b</td><td colspan="2">c</td></tr>
<tr><td>a b a b c</td><td>a b c</td><td>a</td><td>b c</td><td>a</td><td>b c d</td></tr>
<tr><td>ab abc abc abc abcd</td><td>abc abc abcd</td><td>abc</td><td>a | bc | a | bcd</td><td>abcd</td><td>a | bcd | a | bcd | a | bcde</td></tr>
</table>

I. Suppose the above table *continued indefinitely*, and form a new table thus. Suppress the first line (a); suppress a of the second line, and all that it gives rise to in the following lines. Call this the first *derived* table. From this derive another table, and so on continually, the rule being:—To form the $(m+1)^{th}$ derived table, suppress the first line of the m^{th}; then suppress every a of the second line, and all that they give rise to afterwards. The numbers of letters in the successive lines of any derived table are the *differences* of the numbers in the successive lines of the preceding table.

Let A denote either the original, or any derived table; B the next, and S the table consisting of all that part of A, with the exception of its first line, which is suppressed to form B. To each letter (k) in the first line of A corresponds a letter (a) in the first line of S; k gives a group in the second line of A; the corresponding a (the first of this group) gives a *similar* group in the second line of S; and these two groups, being similar, give similar sets of groups in the next lines of A and S respectively; and so on continually. In fact, A and S are, with the exception of their first lines, identical, letter for letter. But the $(p+1)^{th}$ line of A is made up of the p^{th} lines of S and B; whence the proposition.

II. The first line of the m^{th} derived table consists of the same number of letters, arranged in the same numerical groups, as the m^{th} line of the original table. Let the first lines of the m^{th} and $(m+1)^{th}$ derived tables be denoted by P and Q respectively; and assume the truth of the proposition for P. Consider a group of r letters in P; this is a group of $r+1$ with its first letter suppressed; it gives rise to $r-1$ groups of $r+1$,

and a group of $r+2$; and these, with their first letters suppressed, are found in Q. Hence a group of r in P gives $r-1$ groups of r, and a group of $r+1$ in Q. But this is just what arises in the $(m+1)^{\text{th}}$ line of the original table from a group of r in the m^{th} line. In fact, the table consisting of the first lines of the series of derived tables only is an exact copy of the original table, with this exception, that the letters $a, b, c \ldots$ of the latter are everywhere replaced by $b, c, d \ldots$ respectively, in the former.

From this property fn can be calculated (see example). The oblique rows are formed in succession; each value of f obtained is transferred to the left-hand column, giving a new oblique row, ending in a new value of f; and so on to any extent.

n	1	2	3	4	5
f	1	2	5	15	52
Δ	1	3	10	37	
Δ^2	2	7	27		
Δ^3	5	20			
Δ^4	15				

Another process, and an expression for fn, are thus obtained.

If (r, n) denote the number of groups of r in the n^{th} line of the original table, we have evidently $(r+1,\ n+1) = (r,\ n) + r\,.\,(r+1, n)$. Form a table in which the r^{th} number of n^{th} column is $(r+1, n+1)$. The first row is 1, 1, 1, ..., the first column 1, 0, 0, ..., the above relation gives all the rest, and the sum of the n^{th} column is fn. But the above conditions are precisely those for the formation of a table in which the r^{th} number of n^{th} column shall be $\Delta^r 0^n/r!$; (compare the above relation with the following—$\Delta^{(r)} 0^n = \Delta^{(r-1)} 0^{n-1} + r\,.\,\Delta^{(r)} 0^{n-1}$; in which $\Delta^{(r)} 0^n$ stands for $\Delta^r 0^n/r!$) Hence $(r+1, n+1) = \Delta^{(r)} 0^n$, and

$$fn = \Delta 0^n + \Delta^2 0^n/2 + \Delta^3 0^n/3! + \ldots + \Delta^n 0^n/n!;$$

in which the r^{th} term, being $(r+1, n+1)$, is evidently the number of cases with just r distinct endings.

From this expression an algebraic proof may be given of the theorem already derived from elementary considerations; namely, that the n^{th} term of the series $f_1, f_2, f_3 \ldots$ is the same as the n^{th} difference of its first term. We have, Δ and D referring to x and y respectively,

$$D^n\,.\,\Delta^m x^y = \Delta^m\,.\,D^n x^y = \Delta^m\,.\,x^y (x-1)^n;$$

whence $D^n\,.\,\Delta^m x^0 = \Delta^m (x-1)^n$. Now, the above expression may be written without error as an infinite series, and the general term $\Delta^r 0^n/r!$ is also $\Delta^{r-1} 1^{n-1}/(r-1)!$ Hence

$$D^n\,.\,\Delta^r 0^1/r! = D^n\,.\,\Delta^{r-1} 1^0/(r-1)! = \Delta^{r-1} 0^n/(r-1)!;$$

and the theorem follows at once.

It remains to determine the relation between ϕ and f. A line which rhymes with no other may be called an *odd* line. I shall show that the number of arrangements of n lines containing at least one odd line is exactly equal to the number of arrangements of $n+1$ lines which contain no odd line. Take any one of the first set, and add an odd line at the end; then replace the odd lines by as many lines rhyming all together, but not with any other line. The result is one of the second set. Conversely, take any one of the second set, and replace the last line and those which rhyme with it by as many odd lines; then erase the last line. The result is one of the first set. Hence the two sets correspond definitely in pairs, and the number in each is the same; that is, $fn - \phi n = \phi(n+1)$, whence the values of ϕ are derived in succession from those of f. Also $\quad \phi(n+1) = fn - f(n-1) + f(n-2) - \ldots \pm f1.$

Of the two relations, $fn = \Delta^n f1$ and $fn - \phi n = \phi(n+1)$, either may be made a consequence of the other, as follows. It is clear that the number of arrangements of n lines containing k odd lines is, in general, $\phi(n-k) \cdot n(n-1)\ldots(n-k+1)/k!$; $k = n$ gives *one*; $k = n-1$ gives none, but $\phi 1 = 0$. Hence, if $f0 = 1$, we have

$$fn = 1 + n\phi 1 + \tfrac{1}{2}n(n-1)\phi 2 + \ldots + n\phi(n-1) + \phi n,$$

true for $n = 0, 1, 2, \ldots$; which gives $\phi n = \Delta^n f0$, and $\phi n + \phi(n+1) = \Delta^n f1$.

The question of finding fn may be approached somewhat differently, as follows. Of the complete permutations (r^n in number) which can be formed from r given letters taken n (not $< r$) at a time, let the number of those in which *all* appear be denoted by n_r; let P be one of them; and let $ab \ldots$, $hk \ldots$, represent any two of the $r!$ simple permutations of the r letters. If then the result of writing h for every a, k for every b, &c., in P, be called Q; it is evident that P and Q represent exactly the same arrangement of line-endings. (The selection of one out of each such set of $r!$ *equivalent* permutations may be made by requiring that the r letters, so far as the *first entry* of each is concerned, shall present one invariable permutation; compare the notation already used.) Hence $(r+1, n+1) = n_r/r!$, and $n_r = \Delta^r 0^n$. But the last result may be obtained independently; (it is in fact essentially the same as that contained in the first part of Question 8390 [Vol. XLV., p.]; as may be seen by putting letters for persons in that question, and regarding the *places*, 1st, 2nd, 3rd, &c., occupied by the letters as the *things* distributed among them); the formula for fn will then follow, by the above reasoning.

The values of f and ϕ, as far as $n = 14$, are given below.

n	f	ϕ	n	f	ϕ	n	f	ϕ
3	5	1	7	877	162	11	678570	98253
4	15	4	8	4140	715	12	4213597	580317
5	52	11	9	21147	3425	13	27644437	3633280
6	203	41	10	115975	17722	14	190899322	24011157

9413. (J. O'BYRNE CROKE, M.A.)—If D be the distance between the centre of the circumcircle and the point of intersection of the perpendiculars of a triangle, prove that $2D/(1-8\cos A\cos B\cos C)^{\frac{1}{2}} = a/\sin A$.

Solution by D. THOMAS, M.A.; R. KNOWLES, B.A.; *and others.*

Let α, β, γ be the vectors of A, B, C, cooriginating from the circumcentre, so that $T\alpha = T\beta = T\gamma = R$, $S\beta\gamma = -R^2\cos 2A$, &c. The vector of the orthocentre $= \cot B\cot C \cdot \alpha + \ldots + \ldots$, therefore

$$-D^2 = -R^2\left\{\Sigma\cot^2 B\cot^2 C + 2\cot^2 A\cot B\cot C\cos 2A + \ldots + \ldots\right\},$$

$$\therefore\quad D^2 = R^2\left\{(\Sigma\cot B\cot C)^2 - 2\cot^2 A\cot B\cot C\,(1-\cos 2A) + \ldots\right\}$$

$$= R^2\left\{1 - 2\cot A\cot B\cot C\,\Sigma\sin 2A\right\} = \text{the result required.}$$

[By Question 8872, we have $D = R(1-8\cos A\cos B\cos C)^{\frac{1}{2}}$,

hence $\quad 2D/(1-8\cos A\cos B\cos C) = 2R = a/\sin A$, &c.]

9478. (Rev. J. J. MILNE, M.A.)—If p be the sum of the abscissæ, q the sum of the ordinates of two points P, Q of an ellipse; prove that (1) the equation of PQ is $2b^2px + 2a^2qy = b^2p^2 + a^2q^2$; and hence (2) if either (α) p or q be constant, or (β) if p and q be connected by the relation $lp + mq = 1$, the envelope of the line is a parabola.

Solution by R. KNOWLES, B.A.; Prof. A. W. SCOTT, M.A.; *and others.*

If hk be the pole of PQ, its equation is

$$b^2hx + a^2ky = a^2b^2 \qquad (1),$$

$$h = p\,(a^2k^2 + b^2h^2)\,/\,2a^2b^2, \qquad k = q(a^2k^2 + b^2h^2)\,/\,2a^2b^2,$$

$$b^2p^2 + a^2q^2 = 4a^4b^4\,/\,(a^2k^2 + b^2h^2);$$

making these substitutions in (1), we obtain $2b^2px + 2a^2qy = b^2p^2 + a^2q^2$.

(α) If p is constant the equation to the envelope is the parabola $a^2y^2 = b^2p\,(p-2x)$; if q, $b^2x^2 = a^2q\,(q-2y)$.

(β) If $lp + mq = 1$, the envelope is

$$(a^2ly - b^2mx)^2 + 2a^2b^2lx + 2a^2b^2my - a^2b^2 = 0.$$

9439. (A. KAHN, M.A.)—Show, by a general solution, that the roots of $4x^4 + 4x^3 + 13x^2 + 6x + 8 = 0$ are $\frac{1}{4}\{-1 \pm (-7)^{\frac{1}{2}}\}$, $\frac{1}{4}\{-1 \pm (-3)^{\frac{1}{2}}\}$.

Solution by Professor COCHEZ; R. W. D. CHRISTIE; *and others.*

L'équation du 4ᵉ degré $x^4 + ax^3 + bx^2 + cx + d = 0$ peut se mettre sous la forme $\quad (x^2 + \frac{1}{2}ax)^2 + (b - \frac{1}{4}a^2)\,\{x^2 + cx\,/(b - \frac{1}{4}a^2)\} + d = 0,$

et pourra être résolue par les méthodes du second degré dans le cas où

$$c = \tfrac{1}{2}a\,(b^2 - \tfrac{1}{4}a^2).$$

L'équation proposée $4x^4 + 4x^3 + 13x^2 + 6x + 8 = 0$ est dans ce cas; on peut l'écrire $\quad (x^2 + \frac{1}{2}x)^2 + 3\,(x^2 + \frac{1}{2}x) + 2 = 0.$

Posant $x^2 + \frac{1}{2}x = y$, on a à résoudre $y^2 + 3y + 2 = 0$, dont les racines sont -1 et -2. Par suite les quatre valeurs de x seront données par les équations $\quad x^2 + \frac{1}{2}x = -1$ et $x^2 + \frac{1}{2}x = -2$.

[If $p^3 - 4pq + 8r = 0$, any biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$ can be immediately reduced to quadratics; hence the given equation

$$\equiv (x^2 + mx + 2)(x^2 + nx + 1) = 0, \quad \equiv (x^2 + \tfrac{1}{2}x + 2)(x^2 + \tfrac{1}{2}x + 1) = 0.]$$

9437. (H. FORTEY, M.A.)—Show that, if α, β, &c. are the p roots (excluding unity) of $x^{p+1} - mx^p + m - 1 = 0$, the number of ways in which m letters can be arranged n in a row, repetitions being allowed but not more than p consecutive letters being the same, is

$$\frac{m}{(m-1)^2}\,\Sigma\,\frac{(\alpha-1)^2\,\alpha^{n+p}}{\alpha^{p+1} - (p+1)\,\alpha + p}.$$

Solution by Professor SWAMINATHA AIYAR, B.A.

Referring to my solution of Question 9293 (Vol. XLIX., p. 26), let Q_n stand for the required number of ways; of these Q_n ways let those that do not begin with the letter a be q_n in number. Then we have

$$Q_n = q_n + q_{n-1} \ldots + q_{n-p} \quad \text{and} \quad q_n = (m-1)(q_{n-1}+q_{n-2} \ldots + q_{n-p});$$

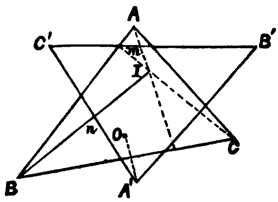
therefore $Q_n = \dfrac{m}{m-1} q_n$. And q_n is the coefficient of x^n in the expansion of

$$\{1-(m-1)(x+x^2 \ldots + x^p)\}^{-1};$$

therefore
$$Q_n = \frac{m}{(m-1)^2} \Sigma \frac{(a-1)^2 a^{n+p}}{a^{p+1}-(p+1)a+p}.$$

8177. (Professor HANUMANTA RAU, M.A.)—The images of the circum-centre of a triangle ABC with respect to the sides are A′, B′, C′; prove that the triangles A′B′C′ and ABC are (1) equal, (2) have the same nine-point circle; also find (3) the equation of the circum-circle of A′B′C′ and the angle at which the two circum-circles cut each other.

Solution by A. GORDON; R. KNOWLES, B.A.; *and others.*



1. OA′ = AI and is parallel to it,
OB′ = BI and is parallel to it,

therefore A′B′ is parallel and equal AB, &c.; therefore the triangle A′B′C′ is equal triangle ABC.

2. I is the circum-centre of A′B′C′ (for IA′ is parallel and equal OA = R = &c.), also IA′ = A′B (each = R), therefore $Bn = nI$; therefore the lines AI, BI, &c. are bisected by B′C′, A′C′, &c., at m, n, &c., and these points are on the nine-point circle of ABC.

But n is also the middle point of A′C′ (for In is a perpendicular from centre I on the chord A′C′), and is therefore a point on the nine-point circle of A′B′C′. These two circles have therefore three points m, n, &c. common, and are therefore coincident.

3. If the mid-point of OI is taken for origin of coordinates, and any rectangular axis for reference, and if $x\cos\alpha + y\sin\alpha - p_1$ is the equation of BC, &c., then the circle about A′B′C′ is

$$\Sigma \sin C\,(x\cos\alpha + y\sin\alpha + p_1)(x\cos\beta + y\sin\beta + p_2) = 0.$$

The equation can also be written in trilinears—

$$R^2(a\sin A+\beta)^2 = \Sigma(\beta\gamma'-\gamma\beta')^2 - 2\Sigma\cos A\,(\gamma\alpha'-\alpha\gamma')(\alpha\beta'-\alpha'\beta),$$

where α', β', γ' are the coordinates of I the orthocentre.

The angle at which the two circum-circles intersect is given by

$$(OI)^2 = R^2 - 2Rr = 2R^2 - 2R^2\cos\phi, \quad \text{or} \quad \cos\phi = \frac{2r+R}{2R}.$$

9217. (Major-General P. O'Connell.)—In using either the French or English Arithmometer, any two numbers each containing less that nine figures can be multiplied together, and the sum of a series each term of which is the product of two such numbers, whether positive or negative, can be obtained without writing down any figures. It is required to find a formula for the product true to, say, thirteen figures on two numbers each of sixteen figures, so that the result may be obtained by the use of the Arithmometer alone, *i.e.*, without intermediate record.

Solution by the Proposer.

Let A and B be two large numbers, and let their digits, counting from left to right, be indicated by numbers written under A and B respectively. Let A_{1-8} mean the first eight highest digits of A, and B_{9-12} the 9th, 10th, 11th, and 12th digits of B; then, if A and B each contain sixteen figures, the following formula will give a result true to thirteen or fourteen figures.

$$\begin{aligned} A \times B = {} & A_{1-8} \times B_{1-8} + A_{1-4} \times B_{9-12} + B_{1-4} \times A_{9-12} + A_{1-2} \times B_{13-14} \\ & + B_{1-2} \times A_{13-14} + A_{5-16} \times B_{9-10} + B_{5-6} \times A_{9-10} \\ & + A_1 \times B_{15} + B_1 \times A_{15} + A_3 \times B_{13} + B_3 \times A_{13} + A_5 \times B_{11} + B_5 \times A_{11} \\ & + A_7 \times B_9 + B_7 \times A_9 \quad \ldots\ldots \quad (1). \end{aligned}$$

If $A = B$, we have

$$\begin{aligned} A^2 = {} & (A_{1-8})^2 + 2\left\{A_{1-4} \times A_{9-12} + A_{1-2} \times A_{13-14} + A_{5-6} \times A_{9-10} \right. \\ & \left. + A_1 \times A_{15} + A_3 \times A_{13} + A_5 \times A_{11} + A_7 \times A_9\right\} \quad \ldots\ldots \quad (2). \end{aligned}$$

In using the second formula, first sum the series under the vinculum, add the result to itself, and finally add $(A_{1\ 8})^2$; by this means A^2 will be obtained true to 13 or 14 figures.

The following formula, to be worked with pen or pencil and paper, will, when its total is added to the above, give a result true to all but the last figure, which may be looked upon as approximative. The dots are to be understood as decimal points.

$$\begin{aligned} \text{Remainder} = 2 \times \left\{A_1 + \cdot A_{16} + A_2 \times \cdot A_{15-16} + A_3 \times \cdot A_{14-16} + A_4 \times \cdot A_{13-15} \right. \\ \left. + A_5 \times \cdot A_{12-14} + A_6 \times A_{11-13} + A_7 \times \cdot A_{10-12} + A_8 \times A_{9-11}\right\} \quad \ldots\ldots \quad (3). \end{aligned}$$

By formula (2), if $A = 3{\cdot}99999\ 99999\ 99999$,

$$A^2 = 15{\cdot}99999\ 99999\ 9867.$$

By formula (3), remainder = 132, giving $A^2 = 15{\cdot}99999\ 99999\ 9999$ for the corrected value.

If $A = 3{\cdot}16227\ 76601\ 68379\ 33 = \sqrt{10}$,

by formula (2), $A^2 = 9{\cdot}99999\ 99999\ 9972$,

by formula (3), remainder $= 27{\cdot}418$, giving

$$A^2 = 9{\cdot}99999\ 99999\ 999948 \text{ instead of } 10.$$

8333. (Professor Hanumanta Rau, M.A.)—Prove that the equations $x^5 + 19x - 140 = 0$, and $7x^4 - 12x^3 + 46x^2 + 12x + 7 = 0$, have two common roots.

Solution by Profs. AIYAR, B.A.; SIRCOM, M.A.; *and others.*

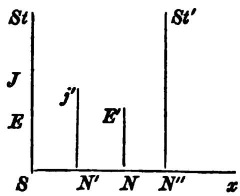
If α, β be two roots of the second equation, the other two roots are evidently $-\dfrac{1}{\alpha}$ and $-\dfrac{1}{\beta}$. Therefore the second equation is reducible to the form $(x^2-px+q)\left(x^2+\dfrac{p}{q}x+\dfrac{1}{q}\right)=0$, and comparing the coefficients we have $p=2$ and $q=7$; and x^2-2x+7 is a factor of $x^5+19x-140$.

9018. (W. J. GREENSTREET, B.A.)—If the Earth and Jupiter are in heliocentric conjunction at the same time as Jupiter and one of his satellites, show that the times when the satellite will appear to an observer to be stationary are the roots of the equation

$$\frac{e^2}{a}+\frac{j^2}{b}+\frac{s^2}{c}+\frac{js}{bc}(b+c)\cos 2\pi\left(\frac{1}{b}-\frac{1}{c}\right)t-\frac{es}{ac}(a+c)\cos 2\pi\left(\frac{1}{a}-\frac{1}{c}\right)t$$
$$-\frac{ej}{ab}(a+b)\cos 2\pi\left(\frac{1}{a}-\frac{1}{b}\right)t=0;$$

where e, j, s are radii of the orbits of the Earth, Jupiter, and the satellite, a, b, c their periodic times, the orbits circular and in one plane.

Solution by W. J. GREENSTREET, B.A.; SARAH MARKS, B.Sc.; *and others.*



Consider motions in two directions perpendicular to each other. After a time t, draw E′N, J′N′, St′N″ perpendicular to Sx.

Then $\mathrm{SN}=e\sin\omega t$.

The position of satellite relative to Earth is given by

$$\left.\begin{array}{l} j\sin\omega_1 t+s\sin\omega_2 t-e\sin\omega t\\ j\cos\omega_1 t+s\cos\omega_2 t-e\cos\omega t\end{array}\right\}.$$

Relative velocities in same directions are

$$\omega_1 j\cos\omega_1 t+\omega_2 s\cos\omega_2 t-\omega e\cos\omega t,\quad -\omega_1 j\sin\omega_1 t-\omega_2 s\sin\omega_2 t+\omega e\sin\omega t.$$

When the satellite is stationary,

$$\frac{j\sin\omega_1 t+s\sin\omega_2 t-e\sin\omega t}{j\cos\omega_1 t+s\cos\omega_2 t-e\cos\omega t}+\frac{\omega_1 j\cos\omega_1 t+\omega_2 s\cos\omega_2 t-\omega e\cos\omega t}{\omega_1 j\sin\omega_1 t+\omega_2 s\sin\omega_2 t-\omega e\sin\omega t}=0;$$

therefore, simplifying, and putting $\omega_1=1/b$, $\omega_2=1/c$, $\omega=1/a$,

$$\omega_1 j^2+\omega_2 s^2+\omega e^2+(\omega_1+\omega_2)js\cos(\omega_1-\omega_2)t-es(\omega+\omega_2)\cos(\omega-\omega_2)t$$
$$-(\omega+\omega_1)je\cos(\omega-\omega_1)t=0,$$

or $$\frac{e^2}{a}+\frac{j^2}{b}+\frac{s^2}{c}+\frac{js}{bc}(b+c)\cos 2\pi\left(\frac{1}{b}-\frac{1}{c}\right)t-\frac{se}{ca}(c+a)\cos 2\pi\left(\frac{1}{c}-\frac{1}{a}\right)t$$
$$-\frac{ej}{ab}(a+b)\cos 2\pi\left(\frac{1}{a}-\frac{1}{b}\right)t=0,\ \text{an equation for } t.$$

9277. (Rev. T. C. SIMMONS, M.A.)—**Prove that the Taylor-circle of a triangle is always greater than its cosine circle, and that in an equilateral triangle the respective areas are in the ratio of 21 to 16.**

Solution by R. F. DAVIS, M.A.

If R be the circum-radius and ω the Brocard-angle, we require to show that $\quad R\left\{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C\right\}^{\frac{1}{2}} > R \tan\omega.$

If $\tan\phi = -\tan A \tan B \tan C$, since

$$\cot A + \cot B + \cot C - \cot A \cot B \cot C = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$$
$$= \text{a positive quantity},$$

$\cot\omega + \cot\phi$ is always positive, $\sin(\phi+\omega)$ is always positive, and $\phi + \omega < 180°$. The expression for the Taylor-circle may be written

$$R\frac{\operatorname{cosec}\phi}{\cot\omega + \cot\phi} = R\frac{\sin\omega}{\sin(\phi+\omega)};$$

hence the above inequality reduces to $R\dfrac{\sin\omega}{\sin(\phi+\omega)}$ is always $> \tan\omega$,

or $\quad \cos\omega > \sin(\omega+\phi)$, $\left[\omega > 0 \text{ and } < 30°\right]$, $\cos^2\omega > \sin^2(\omega+\phi)$.

We may square, as both are positive; therefore $\cos\phi \cdot \cos(\phi + 2\omega) > 0$.

For an *acute*-angled triangle $\cot\phi$ is negative, and ϕ lies between $\frac{1}{2}\pi$ and π, and, since $\phi + 2\omega < \pi + \omega$ ($< 210°$), both cosines are negative, and the inequality holds. When the triangle is *obtuse*-angled, $\cos\phi$ is positive, and ϕ lies between 0 and $\frac{1}{2}\pi$; hence $\cos\phi \cdot \cos(\phi+2\omega) > 0$.

But when the triangle is right-angled $\phi = \frac{1}{2}\pi$, and ω has any value from 0 up to $\cot^{-1} 2$. The radius of Taylor-circle = radius of cosine circle in this case. Therefore it would appear, for a triangle having one angle a little greater than a right angle and the other two angles nearly equal, the above theorem is not true.

When the triangle is equilateral, $\omega = 30°$, $\cot\omega = \sqrt{3}$;

$$\cot\phi = -\left(\frac{1}{\sqrt{3}}\right)^3 = -\frac{1}{3\sqrt{3}} \quad \operatorname{cosec}^2\phi = 1 + \frac{1}{27} = \frac{28}{27},$$

$$\cot\omega + \cot\phi = \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}};$$

and ratios of areas $\qquad \dfrac{\operatorname{cosec}^2\phi}{(\cot\omega + \cot\phi)^2} : \tan^2\omega = 21 : 16.$

8781. (Professor HANUMANTA RAU, M.A.)—If S be the sun, and A and B two planets that appear stationary to one another, show that $\tan SBA : \tan SAB$ = periodic time of A : periodic time of B.

Solution by W. J. GREENSTREET, B.A.; C. BICKERDIKE; *and others.*

By a well-known formula, we have

$$\tan SBA : \tan SAB = \left\{\frac{a^2}{b(a+b)}\right\}^{\frac{1}{2}} : \left\{\frac{b^2}{a(a+b)}\right\}^{\frac{1}{2}} = a^{\frac{3}{2}} : b^{\frac{3}{2}}$$

= (by KEPLER's Law) periodic time of A : periodic time of B.

9044. (S. Tebay, B.A.)—If A be the area of one of the faces of a tetrahedron; X, Y, Z the dihedral angles over A; and

$$M = (1-\cos^2 X-\cos^2 Y-\cos^2 Z-2\cos X\cos Y\cos Z)^{\frac{1}{2}};$$

show that A/M has the same value for all the solid angles.

Solution by the Proposer; Prof. Ignacio Beyens; *and others.*

Let A_1, A_2, A_3, A_4 be the areas of the faces; a, b, c conterminous edges; and α, β, γ the angles contained by bc, ca, ab. Then, from the polar triangle, we have $\cos X+\cos Y\cos Z = \sin Y\sin Z\cos\alpha$.

Squaring and reducing, we find $M = \sin Y\sin Z\sin\alpha$.

Now $$V = \tfrac{2}{3}\cdot\frac{A_1A_3}{b}\sin Y = \tfrac{2}{3}\cdot\frac{A_1A_2}{c}\sin Z;$$

therefore $$V^2 = \tfrac{4}{9}\cdot\frac{A_1{}^2A_2A_3}{bc}\sin Y\sin Z$$

$$= \tfrac{4}{9}\cdot\frac{A_1{}^2A_2A_3}{bc\sin\alpha}\sin Y\sin Z\sin\alpha = \tfrac{4}{9}\cdot\frac{A_1{}^2A_2A_3}{2A_1}M.$$

Therefore $$\frac{A_4}{M} = \tfrac{2}{9}\cdot\frac{A_1A_2A_3A_4}{V^2},$$

which is the same for all the solid angles.

9122. (Professor Hudson, M.A.)—Prove that the locus of the feet of perpendiculars from the vertex of $y^2 = 4ax$ on chords that subtend an angle of 45° at the vertex is $r^2-24ar\cos\theta+16a^2\cos 2\theta = 0$.

Solution by R. Knowles, B.A.; Rev. T Galliers, M.A.; *and others.*

Let PQ be the chord, A the vertex, M the foot of the perpendicular, and x_1y_1, x_2y_2, hk the coordinates of PQ and its pole respectively; the equations to AP, AQ are $x_1y = y_1x$, $x_2y = y_2x$ .... (1, 2).
By the condition, $y_1x_2-x_1y_2 = x_1x_2+y_1y_2$ or $4(k^2-4ah) = (4a+h)^2$...(3).
The equations to AM, PQ are $2ay+kx = 0$, $ky = 2a(x+h)$ ....(4, 5).
From (4) and (5), $k = -2ay/x = -2a\tan\theta$, $h = -(x^2+y^2)/x = -r.\sec\theta$, and substituting these values in (3) we have the result in the question.

8329. (D. Edwardes.)—Prove that (1) the squares of the lengths of the normals drawn from a point xy to the ellipse $b^2x^2+a^2y^2 = a^2b^2$, are given by the equation $\{p^2r^4-(U+p^2V+9q^4)r^2+UV\}^2$
$= 4\{r^4-(2V+3p^2)r^2+3U+V^2\}\{(p^4-3q^4)r^4-(2p^2U-3q^4V)r^2+U^2\}$,
where $U = b^2x^2+a^2y^2-a^2b^2$, $V = x^2+y^2-a^2-b^2$, $p^2=a^2+b^2$, and $q^4=a^2b^2$; and (2) if on the normal at P, a length PQ be measured inwards, equal to the semi-conjugate diameter, the squares of the lengths of the other

three normals drawn from Q are given by the equation

$$(a+b)^2 r^6 - \left\{(a-b)^2 \mathrm{PQ}^2 + 4ab\,(a^2+b^2) - 4a^2b^2 + 4a^4 + 4b^4\right\} r^4$$
$$+ \left\{4\,(a-b)^2 \mathrm{PQ}^2\,(2a^2+2b^2+ab) - 4a^2b^2\,(2a^2+2b^2-7ab)\right\} r^2$$
$$-4\left\{(a-b)^2 \mathrm{PQ}^2 - a^2b^2\right\}^2 = 0.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

1. To eliminate k from the equations

$$\frac{r^2}{k^2} = \frac{x^2}{(k+a^2)^2} + \frac{y^2}{(k+b^2)^2}, \quad \frac{a^2x^2}{(k+a^2)^2} + \frac{b^2y^2}{(k+b^2)^2} = 1.$$

Eliminating x^2 and y^2 alternately, we obtain

$$k^4 + 2b^2k^3 + k^2\left\{b^4 + (a^2-b^2)\,y^2 - a^2r^2\right\} - 2ka^2b^2r^2 - a^2b^4r^2 = 0,$$
$$k^4 + 2a^2k^3 + k^2\left\{a^4 + (b^2-a^2)\,x^2 - b^2r^2\right\} - 2ka^2b^2r^2 - a^4b^2r^2 = 0,$$

whence, introducing U, V, &c., we have

$$2k^3 + k^2\,(r^2 - \mathrm{V}) - q^4r^2 = 0, \quad k^3 + k\,(\mathrm{U} - p^2r^2) - 2q^4r^2 = 0 \;\ldots(1,\ 2),$$

the eliminant of which is the required result.

2. The coordinates of Q will be given by

$$x^2 = \frac{a-b}{a+b}\,(a^2 - \mathrm{PQ}^2), \quad y^2 = \frac{a-b}{a+b}\,(\mathrm{PQ}^2 - b^2),$$

whence $\mathrm{U} = (a-b)^2\,\mathrm{PQ}^2 - a^2b^2$, $\mathrm{V} = -2ab$, then (1) may be written

$$2k^2\,(k+ab) + r^2\,(k^2 - a^2b^2) = 0, \quad 2k^2 + r^2k - abr^2 = 0,$$

and (2) becomes $k^3 + k\left\{(a-b)^2\,\mathrm{PQ}^2 - a^2b^2 - (a^2+b^2)\,r^2\right\} - 2a^2b^2r^2 = 0$,
and the required result at once follows by elimination.

9401. (J. BRILL, M.A.)—Prove that, if n and r be positive integers,

$$\frac{(a+1)(a+2)\ldots(a+n)}{n!} - \frac{(b+1)(b+2)\ldots(b+n)}{(n-1)!} + \frac{(c+1)(c+2)\ldots(c+n)}{(n-2)!\,2!}$$
$$- \frac{(d+1)(d+2)\ldots(d+n)}{(n-3)!\,3!} + \&c. = (r+1)^n,$$

where $a = nr$, $b = (n-1)\,r-1$, $c = (n-2)\,r-2$, $d = (n-3)\,r-3$, &c.

Solution by W. S. FOSTER; SARAH MARKS, B.Sc.; *and others.*

Let $\quad (a+1)(a+2)\ldots(a+n) = a^n + p_1a^{n-1} + \ldots + p_n;$

then the given expression

$$= \Sigma\,\frac{p_q}{n!}\left\{a^{n-q} - n\,.\,b^{n-q} + \frac{n(n-1)}{1\,.\,2}\,c^{n-q} - \ldots\right\} \text{ from } q = 0 \text{ to } q = n,$$

$$= \frac{1}{n!}\,\Sigma\left\{a^{n-q} - n\left[a-(r+1)\right]^{n-q} + \frac{n(n-1)}{1\,.\,2}\left[a-2\,(r+1)\right]^{n-q} - \ldots\right\} p_q.$$

and $\quad a^h - n\left[a-(r+1)\right]^h + \dfrac{n(n-1)}{1\,.\,2}\left[a-2\,(r+1)\right]^h - \ldots$

$$= h!\,.\,\text{coefficient of } x^h \text{ in } e^{ax} - ne^{[a-(r+1)]\,x} + \frac{n(n-1)}{1\,.\,2}\,e^{[a-2(r+1)]\,x} - \ldots,$$

i.e., in $e^{ax}\left[1 - e^{-(r+1)x}\right]^n$, $= 0$, if h is less than n, and $= n!\,(r+1)^n$, if $h = n$;
therefore the given expression $= (r+1)^n$.

9505. (Professor WOLSTENHOLME, M.A., Sc.D.) — Prove, without evolution, or the use of tables, that $3 \times 2^{\frac{2}{3}} - 2^{\frac{1}{3}}$ lies between 3·5022831... and 3·502282... ; the latter being nearer to the exact value.

Solution by D. BIDDLE.

Let $a = 2^{\frac{1}{3}}$. Then $3a^2 - a = x$; also $a^3 = 2$, and $3a^3 - a^2 = ax$, whence $a^2 = 6 - ax$; thus $3(6-ax) - a = x$, and $a(3x+1) = 18 - x$. Cubing both sides, we obtain

$$x^3 + 18x = 106 \quad \text{.................................(1).}$$

We can now find x approximately by a series of trials, correcting according to the successively reduced errors. Let A = the portion of x already found, say 3, which is easily seen to be the first figure, and let $A + z = x$. Then we have $(A+z)^3 + 18(A+z) = 106$, $A^3 + 18A = 106 - k$......(2, 3). Subtracting (3) from (2), we further have

$$3A^2z + 3Az^2 + z^3 + 18z = k, \quad z = k / (3A^2 + 18 + 3Az + z^2) \text{......(4, 5),}$$

or roughly, especially as z diminishes, $z = k / (3A^2 + 18)$ ....(6),

A,	A^3,	18A,	k,	$3A^2 + 18$.
3·	27·	54·	25·	45·
3·5	42·875	63·	0·125	54·75
3·50228	42·95784	63·04104	0·00112	54·7979
3·5022821,	42·9589211,	63·0410778,	0·0000011,	54·79794

3·50228213 nearly $= x$.

[*Otherwise:* If x be the value, and $y = 2x - 7$, we shall then have

$$y + 7 = 6 \times 2^{\frac{2}{3}} - 2^{\frac{4}{3}}, \quad (y+7)^3 = 216 \times 4 - 16 - 18 \times 4\,(y+7),$$

or $(y+7)^3 + 72(y+7) = 848 \equiv y^3 + 21y^2 + 219y + 847$, or $y^3 + 21y^2 + 219y = 1$.

This cubic has only one real root which is positive, and

$$y = \frac{1}{219 + 21y + y^2}, \text{ hence } y < \frac{1}{219}, \text{ and } 21y + y^2 < \frac{1}{10}, \quad \therefore\ y > \frac{1}{219 \cdot 1},$$

$$y + 7 < 7\,\frac{1}{219} > 7\,\frac{1}{219 \cdot 1}, \qquad x < \frac{7}{2} + \frac{1}{438} > \frac{7}{2} + \frac{1}{438 \cdot 2},$$

which are the given limits. Since $21y + y^2$ is nearly $= \frac{1}{10}$, it is clear that the value of x is nearer to the inferior limit.]

"SOMETHING OR NOTHING?" BY CHARLES L. DODGSON, M.A.

In the years 1885, 1886, there appeared in regard to a Solution of Quest. 7695 (see Vol. XLIII., p. 86, and XLIV., p. 24) a discussion about a difficulty in the Theory of Chances, of which the following question was treated as a typical example:—"A random point being taken on a given line, what is the chance of its coinciding with a previously assigned point?" On one side it was maintained that the chance is *absolute zero*: on the other side it was maintained, by myself and others, that it is some sort of *infinitesimal*, and *not* absolute zero. The arguments on both sides were fully stated, and my only excuse, for re-opening the discussion, is

that I have a *new* view of the difficulty to offer to the supporters of the "absolute zero" theory.

I assume that both sides accept the following axioms:—(1) that no aggregate, however infinitely numerous, of *absolute zeroes* can constitute a *magnitude*, however infinitely small; (2) (an example of the preceding) that no aggregate, however infinitely numerous, of *points* can constitute any portion, however infinitely short, of a *line*; and hence (3) that, if the chance of a random point on a line coinciding with *a single selected point* be absolute zero, so also is its chance of coinciding with one or other of *a selected aggregate of points*, however infinitely numerous.

I now propose two questions:—

I. "A random point being taken on a given line, what is the chance of its dividing the line into two *commensurable* parts?" It seems clear that we are here dealing with *a selected aggregate of points*, since it is impossible to mark off any portion of the *line*, and to say "Wherever, in this portion, the random point shall fall, it will divide the whole line into two commensurable parts." I assume, then, that my opponents would answer "It is *absolute zero*."

II. "And what is its chance of dividing the line into two *incommensurable* parts?" Here again they must answer "It is *absolute zero*."

And yet *one or other* of these two events *must* happen! Hence, the sum of the two chances must be mathematically represented by unity; that is, one or other (though we cannot say which) must be — not only "*something*," not only a certain *infinitesimal*, of some inconceivably high order — but must actually reach, if not exceed, the *finite* value of *one-half!*

9506. (Professor HUDSON, M.A.) — Prove that (1) the parabola $y^2 = 2l(x+l)$ can be described by a force to the origin which varies as $r/(x+2l)^3$; and find (2) what ambiguity there is in the case of this law of force.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

The polar equation is $\sin^2\theta = 2\dfrac{l}{r}\cos\theta + 2\dfrac{l^2}{r^2}$,

or
$$2lu = -\cos\theta + (1+\sin^2\theta)^{\frac{1}{2}}, \qquad (u \equiv 1/r),$$

$$2l\left(u + \frac{d^2u}{d\theta^2}\right) = (1+\sin^2\theta)^{\frac{1}{2}} + \frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{(1+\sin^2\theta)^{\frac{1}{2}}}\right)$$

$$= (1+\sin^2\theta)^{\frac{1}{2}} + \frac{\cos 2\theta}{(1+\sin^2\theta)^{\frac{1}{2}}} - \frac{\sin^2\theta\cos^2\theta}{(1+\sin^2\theta)^{\frac{3}{2}}} \equiv \frac{2}{(1+\sin^2\theta)^{\frac{3}{2}}},$$

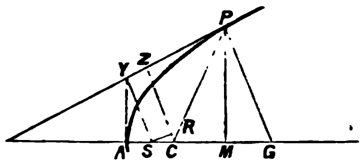
therefore the central force

$$= \frac{h^2}{l}\,u^2/(1+\sin^2\theta)^{\frac{3}{2}} = \frac{h^2u^2}{l}\Big/(2lu+\cos\theta)^3 \equiv \frac{h^2r}{l\,(x+2l)^3}.$$

The centre of force is the centre of curvature at the vertex of the parabola, and, when the moving point reaches the vertex, it can describe, without any extraneous pressure, either the circle of curvature or the other

half of the parabola, apparently at its own choice. Of course the Question is not one of physics at all.

[*Otherwise*:—Let A be the vertex, S the focus, C the centre of force, AS = SC = $\frac{1}{4}l$; draw SY, CZ perpendiculars on the tangent at P, and SR perpendicular on CZ: then AY is perpendicular to AS, and the triangles CRS, ASY are similar;



hence CR : CS = AS : SY,

or $\quad$ CR . SY = AS^2,

and $\quad$ CZ . SY = CR . SY + SY^2 = AS^2 + AS . SP = AS (AS + SG)

$\quad\quad$ = AS (AC + CM + MG) = AS $(x+2l)$,

where PM is the ordinate and PG the normal at P. But we have

central force : normal force = CP : CZ,

$$\text{and normal force} = \frac{(\text{Velocity})^2}{\text{radius of curvature}} = \frac{h^2}{\text{CZ}^2} \cdot \frac{\text{AS}^2}{2\text{SY}^3},$$

$$\text{central force} = \frac{h^2 \cdot \text{CP} \cdot \text{AS}^2}{2\,\text{SY}^3\,\text{CZ}^3} = h^2 . \text{CP} / 2\text{AS}\,(x+2l)^3 = \frac{h^2 r}{l\,(x+2l)^3}.$$

It would be easy to make this strictly geometrical, or rather to frame a geometrical proof on these lines: an expression involving h^2 cannot be made geometrical, but it could readily be proved that the central force must vary directly as CP, and inversely as the cube on AC.]

9087. (H. Fortey, M.A.)—Show that, when the cards are dealt out at whist, the probability that each player holds two or more cards of each suit is ·2062806, &c.; or the odds are about 4 to 1 against the event.

Solution by the Proposer.

If one suit be distributed anyhow amongst 4 persons $\alpha, \beta, \gamma, \delta$, the number of possible distributions is the sum of the coefficients in the expansion of $(\alpha+\beta+\gamma+\delta)^{13}$. But, if each person must have at least 2 cards, the number of distributions is the sum of the coefficients of the terms divisible by $\alpha^2\beta^2\gamma^2\delta^2$; that is, the sum of the coefficients in

$$13!\,\alpha^2\beta^2\gamma^2\delta^2 \left\{ \frac{\Sigma\alpha^5}{7!\,(2!)^3} + \frac{\Sigma\alpha^4\beta}{6!\,3!\,(2!)^2} + \&\text{c.} \right\}.$$

Call this expression M; then, if all the 4 suits are distributed subject to the same restriction, the number of distributions will be the sum of the coefficients in the expansion of M^4. But at the game of whist 13 cards are dealt to each person. Therefore, with the above restriction, the number of possible deals is the coefficient of $\alpha^{13}\beta^{13}\gamma^{13}\delta^{13}$ in M^4, or of $\alpha^5\beta^5\gamma^5\delta^5$ in

$$(13!)^4 \left\{ \frac{\Sigma\alpha^5}{7!\,(2!)^3} + \frac{\Sigma\alpha^4\beta}{6!\,3!\,(2!)^2} + \frac{\Sigma\alpha^3\beta^2}{5!\,4!\,(2!)^2} + \frac{\Sigma\alpha^3\beta\gamma}{5!\,(3!)^2\,2!} + \frac{\Sigma\alpha^2\beta^2\gamma}{(4!)^2\,3!\,2!} + \frac{\Sigma\alpha^2\beta\gamma\delta}{4!\,(3!)^3} \right\}^4,$$

or (reducing) in

$$8580^4\left\{18\,(5)+42\,(41)+63\,(32)+84\,(311)+105\,(221)+140\,(2111)\right\}^4,$$

where for simplicity for $\Sigma\alpha^5$, $\Sigma\alpha^4\beta$, &c., I write (5), (41), &c.

Now, for 18, 42, &c. ... 140 write $a, b, \ldots f$, and let $a\,(5)=\mathrm{A}$, $b\,(41)=\mathrm{B}, \ldots f\,(2111)=\mathrm{F}$. Then (omitting for the present the factor 8580^4), we have to find the coefficient of (5555) in the expansion of

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}+\mathrm{F})^4,$$

or in $$\Sigma\mathrm{A}^4+4\Sigma\mathrm{A}^3\mathrm{B}+6\Sigma\mathrm{A}^2\mathrm{B}^2+12\Sigma\mathrm{A}^2\mathrm{BC}+24\Sigma\mathrm{ABCD}.$$

As a preliminary step I give the squares of the quantities A, B, &c., and the product of every two in terms of the symmetric functions; but as we want ultimately the coefficient of (5555), I omit from these values all functions involving an index greater than 5.

$$\mathrm{A}^2=a^2\,.\,2\,(55),$$
$$\mathrm{B}^2=b^2\left\{2\,(55)+2\,(541)+2\,(442)+4\,(4411)\right\},$$
$$\mathrm{C}^2=c^2\left\{2\,(55)+2\,(532)+2\,(433)+4\,(3322)\right\},$$
$$\mathrm{D}^2=d^2\left\{2\,(442)+4\,(4411)+2\,(4321)+2\,(3322)\right\},$$
$$\mathrm{E}^2=e^2\left\{(442)+2\,(4411)+2\,(433)+2\,(4321)+6\,(4222)+4\,(3322)\right\},$$
$$\mathrm{F}^2=f^2\left\{(4222)+2\,(3322)\right\},$$
$$\mathrm{AB}=ab\,(541),\quad \mathrm{AC}=ac\,(532),\quad \mathrm{AD}=ad\,(5311),\quad \mathrm{AE}=ae\,(5221),\quad \mathrm{AF}=0,$$
$$\mathrm{BC}=bc\left\{2\,(442)+2\,(433)+(4321)\right\},$$
$$\mathrm{BD}=bd\left\{(541)+(532)+2\,(5311)+2\,(4411)+(4321)\right\},$$
$$\mathrm{BE}=be\left\{(532)+(5221)+(4321)+3\,(4222)\right\},$$
$$\mathrm{BF}=bf\left\{(5311)+2\,(5221)\right\},$$
$$\mathrm{CD}=cd\left\{(541)+(5311)+2\,(433)+(4321)+6\,(3331)\right\},$$
$$\mathrm{CE}=ce\left\{(541)+(532)+2\,(5221)+2\,(442)+(4321)+3\,(4222)+2\,(3322)\right\},$$
$$\mathrm{CF}=cf\left\{(5311)+2\,(4411)+(4321)\right\},$$
$$\mathrm{DE}=de\left\{(532)+2\,(5311)+2\,(5221)+(433)+(4321)+3\,(3331)+4\,(3322)\right\},$$
$$\mathrm{DF}=df\left\{(5221)+(4321)+3\,(4222)\right\},$$
$$\mathrm{EF}=ef\left\{(4321)+3\,(3331)+2\,(3322)\right\}.$$

We shall now have no difficulty in finding the coefficient of (5555) in every term of the expansion of $(\mathrm{A}+\mathrm{B}+\mathrm{C}+\&c.)^4$. Take, for instance, the term $12\mathrm{D}^2\mathrm{EF}$; then, omitting for the present the factor 12 which multiplies all terms of that form, we have

$$\begin{aligned}\mathrm{D}^2\mathrm{EF}=\mathrm{DE}\,.\,\mathrm{DF}=d^2ef&\left\{(532)+2\,(5311)+2\,(5221)+(433)+(4321)\right.\\&\left.+3\,(3331)+4\,(3322)\right\}\times\left\{(5221)+(4321)+3\,(4222)\right\}.\end{aligned}$$

But (5555) can arise only from the products of the complementary functions $(5221)\times(433)$, $(4321)(4321)$, $3\,(4222)\times3\,(3331)$ ......(1, 2, 3), and the coefficient is 12 in (1), 24 in (2), and 9×4 in (3); therefore the coefficient of (5555) in $\mathrm{D}^2\mathrm{EF}=d^2ef\,(12+24+36)=72d^2ef$.

It is convenient for the present to divide each coefficient by 24, which factor can be re-introduced after summation, and this being understood the coefficient of (5555) in $\mathrm{D}^2\mathrm{EF}$ is $3d^2ef$. Determining the other co-

efficients in like manner, and collecting those corresponding to ΣA^4, ΣA^3B, &c., we get

Group of terms.	Coefficient of (5555).
ΣA^4	$a^4 + 9(b^4 + c^4 + d^4 + e^4) + f^4 = \text{P}$, suppose,
ΣA^3B	$b^3(2a + 6d + f) + c^3(2a + 6e) + d^3(a + 6b + 3c + 6e + 2f)$
	$+ e^3(a + 3b + 6c + 12d + 7f) + f^3\,\frac{3}{2}e = \text{Q}$,
ΣA^2B^2	$a^2(b^2 + c^2) + b^2(c^2 + 4d^2 + 2e^2) + c^2(2d^2 + 4e^2 + 2f^2)$
	$+ d^2(8e^2 + f^2) + 2e^2f^2 = \text{R}$,
ΣA^2BC	$b^2(ad + 3cd + 2ce + 3cf + 2de)$
	$+ c^2(ae + 2bd + 3be + 2bf + 8de + df + 2ef)$
	$+ d^2(2bc + 2be + bf + 3ce + 5cf + 3ef)$
	$+ e^2(\frac{1}{2}ad + 2bc + 4bd + \frac{5}{2}bf + \frac{17}{2}cd + \frac{7}{2}cf + 3df) + f^2(cd + ce + \frac{5}{2}de) = \text{S}$,
$\Sigma ABCD$	$abc(d + e) + acde + bc(5de + 2df + ef) + def(b + c) = \text{T}$.

Substituting for a, b, c, &c., the numerical values 18, 42, 63, &c., and reducing, it will be found that

$$\text{P} = 2{,}096{,}086{,}738, \quad \text{Q} = 4{,}366{,}141{,}668, \quad \text{R} = 1{,}676{,}987{,}172,$$
$$\text{S} = 3{,}888{,}459{,}288, \quad \text{T} = 366{,}476{,}292.$$

Therefore the coefficient of (5555) in $(A + B + C + \&c.)^4$, or in

$$\Sigma A^4 + 4\Sigma A^3B + \&c. = \text{P} + 4\text{Q} + 6\text{R} + 12\text{S} + 24\text{T} = 85{,}079{,}518{,}906.$$

Now, remembering that every coefficient was divided by 24, and that we have omitted the factor 8580^4, we see that the number of ways of dealing out the cards so that each player may hold 2 or more of each suit,

$$= 24 \times 8580^4 \times 85{,}079{,}518{,}906 = \text{H suppose.}$$

Then $$\log \text{H} = 28{\cdot}0439855.$$

Let K = number of ways of dealing out 4 hands, without any restriction. Then $\text{K} = 52! \div (13!)^4$ and $\log \text{K} = 28{\cdot}7295271$, therefore

$$\log \text{H/K} = \log \text{H} - \log \text{K} = {\cdot}3144584 = \log {\cdot}2062806,$$

therefore $\quad \text{H/K} = {\cdot}2062806 \ldots =$ the required chance,

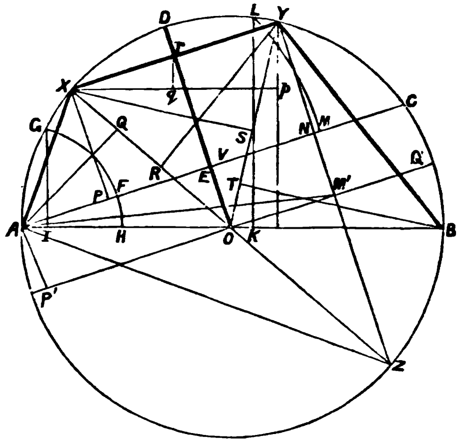
and the odds are about 4 to 1 against the event.

9481. (W. S. McCay, M.A.)—AB is the diameter of a semicircle; show how to draw a chord XY in a given direction, so that the area of the quadrilateral AXYB may be a maximum.

Solutions by (1) *the* Proposer; (2) D. Biddle *and* Prof. MacMahon.

1. Drawing the chord XX′ perpendicular to XY, the quadrilateral is equal to the triangle XX′B, and if the chord BC be drawn parallel to XX′ the problem is reduced to construct the maximum rectangle standing on BC and having its vertices (XX′) on the circle, the solution of which is well known; X is the middle point of the intercept on the tangent at X between BC and its perpendicular bisector (Townsend, Vol. i., p. 47).

2. Let O be the mid-point of AB, and centre of the semicircle. Draw AC parallel to the given direction, and OD perpendicular to it, cutting AC in E. Bisect AE in F, through which, with centre A, draw the arc GFH. Also draw GI perpendicular to AB, and make IK = AO. Draw KL at right angles to AB, and make AM = AL. Bisect FM in V, and make EP = VF. Finally draw PX parallel to OD, and XY parallel to AC. The quadrilateral AXYB, being completed, is that required.



For the conditions are fulfilled when, with infinitesimal bases tangential to the semicircle at X, Y, the pairs of triangles whose apices are respectively at A, Y, and at B, X, counterbalance each other, that is, when YT − YS = XR − XQ. Let AB = 1, then YS = XR = $\frac{1}{2}(1-\cos \mathrm{XOY})$;
also XQ = $\frac{1}{2}(1-\cos \mathrm{AOX})$, and YT = $\frac{1}{2}(1-\cos \mathrm{BOY})$;
whence $\frac{1}{2}(\cos \mathrm{AOX}+\cos \mathrm{BOY}) = \cos \mathrm{XOY} = 1-2\sin^2 \mathrm{DOX}$.
But it is easy to see, by reference to *p*, *q*, *r* in the diagram, that

$$\tfrac{1}{2}(\cos \mathrm{AOX}+\cos \mathrm{BOY}) : \sin \mathrm{DOX} = \mathrm{AE} : \mathrm{AO}.$$

We therefore have a quadratic, whence

$$\sin \mathrm{DOX} = \left\{(\tfrac{1}{2}\mathrm{AE})^2+\tfrac{1}{2}\right\}^{\frac{1}{2}}-\tfrac{1}{2}\mathrm{AE}.$$

The construction follows this formula.

[*Otherwise:*—For the maximum area, the squares of the variable sides must be in Arithmetical Progression. For, if we take X′Y′ a consecutive position of XY, then it is evident that the areas XX′Y + YY′X differ only by an infinitesimal of the second order from the areas AXX′ + BYY′; hence, in the limit, $(\mathrm{AX})^2+(\mathrm{BY})^2 = 2(\mathrm{XY})^2$. Now take, further, Z the diametral point of X; draw AZ, YZ; draw the diameter OM parallel to the given direction and hence bisecting YZ at right angles, and draw AP′ perpendicular to OP′. Then $\mathrm{AY}^2+\mathrm{AZ}^2 = 2\mathrm{YZ}^2 = 8\mathrm{M'Y}^2$; hence $\mathrm{AM'}^2 = 3\mathrm{M'Y}^2$, $\mathrm{AO}^2+\mathrm{OM'}^2+2\mathrm{PO}.\mathrm{OM'} = 3(\mathrm{OY}^2-\mathrm{OM'}^2)$, $(\mathrm{P'O}+2\mathrm{OM'})(2\mathrm{OM'}) = 2\mathrm{OA}^2$; and thus the quadrilateral may be constructed by finding Q′, so that $\mathrm{P'Q'}.\mathrm{OQ'} = 2\mathrm{OA}^2$, and through M′, the mid-point of OQ′, drawing M′Y perpendicular to OM′; this determines the vertices Y and X.]

9459. (Professor GENESE, M.A.)—If ρ, θ be the polar coordinates of a point whose coordinates referred to axes inclined at any angle w are x, y, then x/ρ, y/ρ may be denoted by C (θ), S (θ). Prove that

$$S(\theta-\phi) = S(\theta) \,.\, C(\phi) - C(\theta) \,.\, S(\phi),$$
$$C(\theta+\phi) = C(\theta) \,.\, C(\phi) - S(\theta) \,.\, S(\phi).$$

Solution by Profs. MACMAHON, B.A., IGNACIO BEYENS; *and others.*

The following is a general proof by the method of projections. As $\cos\theta$ is the orthogonal projecting factor, so C (θ) may be called the ω-gonal (omégonal) projecting factor. Take any two angles θ and ϕ, positive or negative, the initial line of ϕ being the terminal line of θ; then a unit distance on the terminal line of ϕ resolves into C (ϕ) on its initial line and into S (ϕ) in the ω-gonal direction which makes angle $\omega+\theta$ with x-axis. Take the ω-gonal projection of all three on this axis, then (whatever θ and ϕ) $C(\theta+\phi) = C(\phi) \,.\, C(\theta) + S(\phi) \,.\, C(\omega+\theta)$; but it is evident from a figure that $C(\omega+\theta) = -S(\theta)$, therefore

$$C(\theta+\phi) = C(\theta) \,.\, C(\phi) - S(\theta) \,.\, S(\phi) \quad\text{.....................(1).}$$

Now change θ into $(\omega-\theta)$, and observe that

$$S(\omega-\theta) = C(\theta), \quad C(\omega-\theta) = S(\theta),$$

therefore $$S(\theta-\phi) = S(\theta) \,.\, C(\phi) - C(\theta) \,.\, S(\phi) \quad\text{.....................(2);}$$

hence also $$S(\theta+\phi) = S(\theta) \,.\, C(-\phi) + C(\theta) \,.\, S(\phi) \quad\text{..................(3),}$$
$$C(\theta-\phi) = C(\theta) \,.\, C(-\phi) + S(\theta) \,.\, S(\phi) \quad\text{..................(4).}$$

It is worthy of remark that the orthogonal formulas for $\sin(\theta+\phi)$ and $\cos(\theta-\phi)$ are not so *general* in their nature as the other two formulas, since they require $C(-\phi) = C(\phi)$, which is true only when $\phi = n\pi$ (n an integer). It may also be noted that, when $\phi = n \,.\, \frac{1}{2}\pi$, ($n$ an odd integer), $C(-\phi) = -C(\phi)$.

9477. (SWIFT P. JOHNSON, M.A.)—A, B, C and a, b, c are two triads of points on a sphere; show that, if the circumcircles of the triangles Abc, Bca, Cab meet in a point, then the circumcircles of the triangles aBC, bCA, cAB will also meet in a point.

Solution by Professor SCHOUTE.

A stereographic transformation and a transformation by reciprocal radii vectores, the centrum of which is the point common to the circles Abc, Bca, Cab, lead to the following self-evident problem. When on the sides $b'c'$, $c'a'$, $a'b'$ the points A′, B′, C′ are given, the circles a'B′C′, b'C′A′, c'A′B′ meet in a point.

9516. (D. BIDDLE.)—Prove or disprove that (1) a circle B is not properly drawn at random within a given circle A, unless its centre be first taken at random on the surface of A, and its radius be subsequently

taken at random within the limits allowed by the position of its centre; (2) putting unity for the radius of A, r for the radius of B, and x for the distance between the two centres, there are two things requisite in order that B may include the centre of A, namely, that x be less than $\frac{1}{2}$, and that r be between x and $1-x$; (3) from a favourably placed centre, the chance of the radius of B being such as to make it include the centre of A is $(1-2x)/(1-x)$; (4) the chance is identical for $2\pi x\,.\,dx$ positions, which form the circumference of a circle of radius x, around the centre of A; (5) the probability that a circle B, drawn at random in a given circle A, shall include the centre of A, is *not* correctly found by the formula

$$P = 2\pi \int_0^{\frac{1}{2}} \int_x^{1-x} x\,dx\,dy + 2\pi \int_0^1 \int_0^{1-x} x\,dx\,dy = \tfrac{1}{4},$$

since this assumes that the number of circles capable of being drawn from any centre is proportioned to the upper limit of the radius; leaves out of account that *one* centre, *one* radius, *one* circle B, are taken each time; and gives a result which actually does not fall short of the chance that the centre alone shall be favourably placed; (6) the probability in the case referred to is correctly found as follows:—

$$P = 2\pi \int_0^{\frac{1}{2}} x \left(\frac{1-2x}{1-x}\right) dx + 2\pi \int_0^1 x\,.\,dx = 1\tfrac{1}{2} + 2\log_e \tfrac{1}{2}$$
$$+\,\bar{2}\cdot61370564 = 0\cdot11370564, \text{ or less than } \tfrac{1}{8}.$$

Solution by W. S. B. Woolhouse, F.R.A.S.

By "drawn at random," as stated at the beginning of this question, it should be understood that there is not to be any condition whatever affecting the circle B, excepting that it must be wholly included within the circle A. In order that the circle B may be properly regarded as so drawn at random, the correct mode of procedure is to discuss all the cases that can arise from an indiscriminate admission of every position of the centre, and also, at the same time, of every possible magnitude of the radius consistent with the foregoing condition alone. This important principle is strictly carried out in (5), the first working stated in the question by which I consider the probability $P = \frac{1}{4}$ to be correctly obtained.

The probability otherwise deduced in (6) would be correct if the circle B were drawn at random in a very restricted sense, that is, assuming (1) that its centre is first taken at random within the circle A, and (2) that *only one* radius is allowable. This is undoubtedly special, and the circle B is not drawn at random in the free sense of the term, the cases taken into consideration being but an infinitesimal portion of the total cases.

To further show how misleading partial considerations are in questions of this nature, suppose that the circle B is first taken at random from an infinite set of circles having radii from 0 to 1, and then placed only once on the circle A, radius = 1. Now, if the radius ρ of B fall between 0 and $\frac{1}{2}$, it may or may not include the centre of A; and if it should fall between $\frac{1}{2}$ and 1, it will, when placed on A, be certain to include the centre. The resulting probability according to this arrangement will therefore exceed $\frac{1}{2}$. Thus, when $\rho = 0 \ldots \frac{1}{2}$, the probability will be

$$2\pi\rho^2 / 2\pi(1-\rho)^2 = \rho^2/(1-\rho)^2;$$

and, when $\rho = \frac{1}{2} \dots 1$, it will be unity. The resulting probability is therefore $$\int_0^{\frac{1}{2}} \frac{\rho^2}{(1-\rho)^2}\,d\rho + \int_{\frac{1}{2}}^1 d\rho = 2(1-\log_e 2) = \cdot 6121.$$

If the general problem be considered absolutely, so as to include all possible cases of the circle B and its positions, then, when the radius ρ ranges from 0 to $\frac{1}{2}$, the successful positions of the centre $= 2\pi\rho^2$; and when ρ ranges from $\frac{1}{2}$ to 1, the number of positions $= 2\pi(1-\rho)^2$, all of which are successful. Whence the total successful positions

$$= 2\pi \int_{\frac{1}{2}}^1 \rho^2\,d\rho + 2\pi \int_{\frac{1}{2}}^1 (1-\rho)^2\,d\rho = 2\pi\left(\tfrac{1}{24} + \tfrac{1}{24}\right) = \tfrac{1}{6}\pi.$$

Also for every value of ρ the number of positions $= 2\pi(1-\rho)^2$, so that the total number of positions $= 2\pi \int_0^1 (1-\rho)^2\,d\rho = \frac{2}{3}\pi$. And therefore by division $P = \frac{1}{4}$, as before, which is the true result.

9407. (W. J. Greenstreet, B.A.)—From a point outside a circle centre C, APQ is drawn cutting it in P and Q; AT is a tangent at T: show that it is always possible to draw such a line that AP shall equal PQ, as long as AC $\not>$ 3CT; and that then $3\cos TAC = 2\sqrt{2}\cos PAC$.

Solution by R. Knowles, B.A.; Sarah Marks, B.Sc.; *and others.*

PQ cannot be greater than the diameter of the circle; it may be equal to it, and in that case AC = 3 times the radius.

$\cos TAC = AT/AC$, $\cos PAC = AN/AC$ (N the mid-point of PQ), therefore $\cos TAC / \cos PAC = AT/AN$. $AN = \frac{3}{2}AP$; and (Euc. III. 36) $AT = \sqrt{2}\,AP$, therefore $3\cos TAC = 2\sqrt{2}\cos PAC$.

9425. (Professor Hanumanta Rau, B.A.)—Prove that the sum of the products of the first n natural numbers taken three at a time is
$$\tfrac{1}{48}n^2(n+1)^2(n-1)(n-2).$$

Solution by Prof. Aiyar, B.A.; Rosa Whapham; *and others.*

Let ${}_nP_3$ stand for the sum of the products taken three at a time. Then

$${}_nP_3 = {}_{n-1}P_3 + n\,{}_{n-1}P_2 = {}_{n-1}P_3 + n \cdot \tfrac{1}{2}\left\{\frac{(n-1)^2 n^2}{4} - \frac{(n-1)\,n\,(2n-1)}{6}\right\}$$
$$= {}_{n-1}P_3 + 15\frac{n^{(5)}}{5!} + 20\frac{n^{(4)}}{4!} + 6\frac{n^{(3)}}{3!}.$$

Therefore $${}_nP_3 = \Sigma n\,{}_{n-1}P_2 = 15\frac{(n+1)^{(6)}}{6!} + 20\frac{(n+1)^{(5)}}{5!} + 6\frac{(n+1)^{(4)}}{4}$$
$$= \tfrac{1}{48}n^2(n+1)^2(n-1)(n-2).$$

Similarly,

$$_n\mathrm{P}_4 = \Sigma n\, {}^{n-1}\mathrm{P}_3 = 105\,\frac{(n+1)^{(8)}}{8\,!} + 210\,\frac{(n+1)^{(7)}}{7\,!} + 130\,\frac{(n+1)^{(6)}}{6\,!} + 24\,\frac{(n+1)^{(5)}}{5\,!}.$$

And, calling the coefficients (105, 210 &c.), a, b, c, d, we have

$$_n\mathrm{P}_5 = 9a\,\frac{(n+1)^{(10)}}{10\,!} + 8\,(a+b)\,\frac{(n+1)^{(9)}}{9\,!} + 7\,(b+c)\,\frac{(n+1)^{(8)}}{8\,!}$$
$$+ 6\,(c+d)\,\frac{(n+1)^{(7)}}{7\,!} + 5\,.\,d\,\frac{(n+1)^{(6)}}{6\,!};$$

and so on for $_n\mathrm{P}_6$, $_n\mathrm{P}_7$, &c.; the first coefficient in $_n\mathrm{P}_r$ being $1\,.\,3\,.\,5\,\ldots\,(2r-1)$.

9390. (N'IMPORTE.)—In any triangle ABC, prove that

$$a\cos 2\mathrm{A}\cos(\mathrm{B}-\mathrm{C}) + \&\text{c.} = -\frac{2\Delta}{\mathrm{R}} = -\frac{8\Delta^2}{abc}.$$

Solution by G. G. STORR, M.A.; J. YOUNG, M.A.; *and others.*

$$a\cos 2\mathrm{A}\cos(\mathrm{B}-\mathrm{C}) = 2\mathrm{R}\sin\mathrm{A}\cos 2\mathrm{A}\cos(\mathrm{B}-\mathrm{C})$$
$$= \mathrm{R}\,(\sin 3\mathrm{A} - \sin\mathrm{A})\cos(\mathrm{B}-\mathrm{C});$$
$$\Sigma\sin 3\mathrm{A}\cos(\mathrm{B}-\mathrm{C}) = 0,$$
$$\Sigma\sin\mathrm{A}\cos(\mathrm{B}-\mathrm{C}) = \sin 2\mathrm{A} + \sin 2\mathrm{B} + \sin 2\mathrm{C} = 2\Delta/\mathrm{R}^2;\ \text{hence, \&c.}$$

9469. (W. J. C. SHARP, M.A.)—If p be a prime number and $r < p-1$, prove that (1) $r!\,(p-r-1)! + (-1)^r$ is a multiple of p; and hence (2), if $p = 2q-1$, $\{(q-1)!\}^2 + (-1)^{q-1}$ is a multiple of $2q-1$.

Solution by W. S. FOSTER.

1. Since $(p-r-1)! = \mathrm{M}(p) + (-1)^{p-r-1}(r+1)(r+2)\ldots(p-1)$, therefore $r!\,(p-r-1)! = \mathrm{M}(p) + (-1)^{p-r-1}(p-1)!$; and, since p is a prime number, we have

$(p-1)! + 1 = \mathrm{M}(p)$, therefore $r!\,(p-r-1)! = \mathrm{M}(p) - (-1)^{p-r-1}$;

and p must be greater than 2, or r would be nothing, therefore $p-1$ must be even, therefore $(-1)^{p-r-1} = (-1)^r$,

therefore $r!\,(p-r-1)! + (-1)^r = \mathrm{M}(p)$.

2. Let $p = 2q-1$, and $r = q-1$, therefore $\{(q-1)!\}^2 + (-1)^{q-1}$ is a multiple of $2q-1$.

9365. (W. J. BARTON, M.A.)—In the expansion of $(1-3x+3x^2)^{-1}$ show that the coefficient of x^{6n-1} is zero.

Solution by H. FORTEY, M.A.; C. E. WILLIAMS, M.A.; *and others.*

Let

$$(1-3x+3x^2)^{-1} = (1-\alpha x)(1-\beta x) = (1+\alpha x+\alpha^2x^2+\&c.+\alpha^{6n-1}x^{6n-1}+\&c.)$$
$$\times(1+\beta x+\beta^2x^2+\&c.+\beta^{6n-1}x^{6n-1}+\&c.).$$

Here the coefficient of $x^{6n-1} = \alpha^{6n-1}+\alpha^{6n-2}\beta+\&c.+\alpha\beta^{6n-2}+\beta^{6n-1}$
$$= (\alpha^{6n}-\beta^{6n})/(\alpha-\beta).$$

Now $\alpha+\beta=3$, $\alpha\beta=3$; therefore $\alpha = \frac{1}{2}[3+(-3)]^{\frac{1}{2}}$, $\beta = \frac{1}{2}[3-(-3)]^{\frac{1}{2}}$,
$$\alpha^2 = \tfrac{3}{2}[1+(-3)]^{\frac{1}{2}} = 3\omega, \quad \beta^2 = \tfrac{3}{2}[1-(-3)]^{\frac{1}{2}} = 3\omega^2,$$
where ω is a cube root of unity, therefore $\alpha^6 = 3^3 = \beta^6$, and $\alpha^{6n}-\beta^{6n} = 0$.

[We know that, when $b^2 < 4ac$, the coefficient of x^n in the expansion of

$(ax^2+bx+c)^{-1}$ is $$\frac{r^n}{c}\cdot\frac{\sin(n+1)\gamma}{\sin\gamma},$$

where $r\cos\gamma = -b/2c$, and $r\sin\gamma = (4ac-b^2)^{\frac{1}{2}}/(2c)$; hence, with the given values, we get for the coefficient of x^{6n-1}

$$r^{\frac{1}{2}(6n-1)}\cdot\frac{\sin 6n\gamma}{\sin\gamma} = \frac{r^{\frac{1}{2}(6n-1)}\sin 180x}{\sin 30^\circ} = 0.]$$

The readiest way of obtaining the development is perhaps by HORNER'S method of *Synthetic Division*; and by observing the law of the series, we see that if A, D be two consecutive coefficients, these and the following coefficients will be A, 0, -3A, -9A, -18A, -27A, -27A, 0, &c.; hence, if 0 occur in any term, it will occur every 6th term. But it occurs in the 6th term, and therefore occurs in the $6n$th term, that is to say, as the coefficient of x^{6n-1}.

9503. (Professor BORDAGE.)—Show that the roots of the equation $2^{2x+2}+4^{1-x} = 17$ are $x = \pm 1$.

Solution by A. M. WILLIAMS, M.A.; A. H. LEWIS; *and others.*

The equation becomes $$2^{2x}+\frac{1}{2^x} = \tfrac{17}{4}.$$

Solving in the usual way, we get $2^{2x} = 2^2$, or 2^{-2}.

9389. (Professor HANUMANTA RAU, M.A.)—Prove (1) that $\sin 6^\circ$ is a root of the equation $16x^4+8x^3-16x^2-8x+1 = 0$;
and (2) express the remaining roots in terms of trigonometrical functions.

Solution by J. YOUNG, M.A.; Professor NASH; *and others.*

The equation $2\sin 5\theta - 1 = 0$ has $\theta = 6^\circ$ for one solution, and $\theta = 30^\circ$ for another; expanding and removing the factor $2\sin\theta - 1$, we have the equation given in the question, on writing x for $\sin\theta$. The remaining roots will be found to be $\sin 78^\circ$, $\sin 222^\circ$, $\sin 246^\circ$; or, in terms of functions of acute angles, the four roots are $\sin 6^\circ$, $\cos 12^\circ$, $-\cos 24^\circ$, $-\cos 48^\circ$.

9410. (A. E. THOMAS.)—If n and r are positive integers, and $n > r$, then (e being the Naperian base)

$$1+\frac{n+1}{r+1}+\frac{1}{2!}\cdot\frac{(n+1)(n+2)}{(r+1)(r+2)}+\frac{1}{3!}\,\frac{(n+1)(n+2)(n+3)}{(r+1)(r+2)(r+3)}+\dots\text{etc. } ad\ inf.$$

$$=e\left\{1+\frac{n-r}{r+1}+\frac{1}{2!}\,\frac{(n-r)(n-r-1)}{(r+1)(r+2)}+\frac{1}{3!}\,\frac{(n-r)(n-r-1)(n-r-2)}{(r+1)(r+2)(r+3)}+\dots\text{etc.}\right\}$$

Solution by R. F. DAVIS, M.A.

This identity follows from equating the coefficients of x^{n-r} in the development of $(1+x)^n e^{1+x} \equiv e\left\{(1+x)^n e^x\right\}$, and dividing each side by ${}_nC_r$; and the necessity for $n > r$ is easily seen.

9499. (Professor ATH BIJAH BHUT.)—Prove that the orthocentre of a triangle is the centroid of three weights, proportional to tan A, tan B tan C, placed at the corners A, B, C.

Solution by W. J. GREENSTREET, B.A.; Col. H. W. L. HIME; *and others.*

With trilinear notation, $\bar{a}$, $\bar{\beta}$, $\bar{\gamma}$ being the centroid of masses $a\alpha$, $b\beta$, $c\gamma$ or $\alpha \sin A$, $\beta \sin B$, $\gamma \sin C$, at the angular points, we have

$$\bar{a} = a\alpha\, b \sin c \,/\, (a\alpha + b\beta + c\gamma) = \alpha \,.\, 2\Delta/2\Delta = \alpha.$$

Similarly, $\quad \bar{\beta} = \beta$ and $\bar{\gamma} = \gamma$.

Now, the orthocentre is sec A, sec B, sec C, therefore it is the centroid of masses sin A, sec A, etc., or tan A, tan B, tan C.

[The in-centre is 1, 1, 1, therefore is centroid of masses sin A, sin B, sin C; the circum-centre is cos A, cos B, cos C, therefore is centroid of masses sin 2A, sin 2B, sin 2C; the ex-centre (A) is -1, 1, 1, therefore is centroid of masses $-\sin A$, sin B, sin C; the nine-point centre is cos (B$-$C), etc., therefore is centroid of masses sin A cos (B$-$C), etc., *i.e.*, of masses sin 2B + sin 2C, etc.; the symmedian-point is sin A, sin B, sin C, therefore is centroid of masses $\sin^2 A$, $\sin^2 B$, $\sin^2 C$; the centroid of the triangle is cosec A, etc., therefore is centroid of masses 1, 1, 1.]

9482. (S. TEBAY, B.A.)—AB, AC, AD are edges of a tetrahedron; BE, CF, DG perpendiculars on the opposite faces; P, Q, R their areas; p, q, r the areas CED, DFB, BGC; and S the area of the base BCD; prove that $Pp + Qq + Rr = S^2$.

Solution by W. S. FOSTER; Professor BEYENS; *and others.*

Let vectors AB, AC, AD be α, β, γ respectively; and let $xV\alpha\beta$ be the vector DG; then, since G is in the plane ABC,

$$S\alpha\beta\,(\gamma + xV\alpha\beta) = 0, \quad \text{therefore} \quad x\,(V\alpha\beta)^2 = -S\alpha\beta\gamma.$$

Now $\quad r : \mathrm{R} = $ tetrahedron DBCG : tetrahedron DBCA

$$= \mathrm{S}(\alpha-\gamma)(\beta-\gamma)\,x\mathrm{V}\alpha\beta : \mathrm{S}(\alpha-\gamma)(\beta-\gamma)(-\gamma)$$

$$= x\mathrm{S}(\alpha\beta-\alpha\gamma-\gamma\beta)\,\mathrm{V}\alpha\beta : -\mathrm{S}\alpha\beta\gamma,$$

and $$\mathrm{R}^2 = -\tfrac{1}{4}(\mathrm{V}\alpha\beta)^2,$$

therefore $\mathrm{R}r : -\tfrac{1}{4}(\mathrm{V}\alpha\beta)^2 = (\mathrm{V}\alpha\beta)^{-2}\mathrm{S}(\mathrm{V}\alpha\beta+\mathrm{V}\gamma\alpha+\mathrm{V}\beta\gamma)\,\mathrm{V}\alpha\beta : 1$;

therefore $$\mathrm{R}r = -\tfrac{1}{4}\mathrm{S}(\mathrm{V}\alpha\beta+\mathrm{V}\beta\gamma+\mathrm{V}\gamma\alpha)\,\mathrm{V}\alpha\beta.$$

Similarly, $$\mathrm{Q}q = -\tfrac{1}{4}\mathrm{S}(\mathrm{V}\alpha\beta+\mathrm{V}\beta\gamma+\mathrm{V}\gamma\alpha)\,\mathrm{V}\gamma\alpha,$$

and $$\mathrm{P}p = -\tfrac{1}{4}\mathrm{S}(\mathrm{V}\alpha\beta+\mathrm{V}\beta\gamma+\mathrm{V}\gamma\alpha)\,\mathrm{V}\beta\gamma,$$

therefore $$\mathrm{P}p+\mathrm{Q}q+\mathrm{R}r = -\tfrac{1}{4}(\mathrm{V}\alpha\beta+\mathrm{V}\beta\gamma+\mathrm{V}\gamma\alpha)^2$$

$$= \tfrac{1}{4}\left(\frac{\mathrm{S}\alpha\beta\gamma}{\mathrm{AH}}\right)^2 \text{ if AH be the perpendicular on BCD}$$

$$= \left(\frac{6 \text{ vol. ABCD}}{2\mathrm{AH}}\right)^2 = \left(\frac{3 \text{ vol. ABCD}}{\mathrm{AH}}\right)^2 = (\text{triangle BCD})^2.$$

[The theorem has been otherwise proved by ordinary methods.]

8941. (W. J. C. Sharp, M.A.)—Prove that the conditions that the binary quantic $(a, b, c \ldots \between x, y)^n$ should be reducible to a binomial form, are

$$\left\| \begin{matrix} a, & b, & c, & d \ldots \\ b, & c, & d, & e \ldots \\ c, & d, & e, & f \ldots \end{matrix} \right\| = 0.$$

[This is a generalisation of the catalecticant of the quartic; those of quantics of higher order admit of similar extension.]

Solution by D. Edwardes.

Let the factors of the binomial form be given by $\mathrm{A}x^2+\mathrm{B}xy+\mathrm{C}y^2$. Then, substituting differentiating symbols for the variables and operating on the quantic, the result must vanish identically. We thus get

$$\left.\begin{matrix} \mathrm{A}c-\mathrm{B}b+\mathrm{C}a = 0 \\ \mathrm{A}d-\mathrm{B}c+\mathrm{C}b = 0 \\ \mathrm{A}e-\mathrm{B}d+\mathrm{C}c = 0 \\ \mathrm{A}f-\mathrm{B}e+\mathrm{C}d = 0 \\ \&\mathrm{c}. \end{matrix}\right\} \text{ or } \left\| \begin{matrix} a, & b, & c, & d \ldots \\ b, & c, & d, & e \ldots \\ c, & d, & e, & f \ldots \end{matrix} \right\| = 0.$$

9511. (E. B. Elliott, M.A.)—Of inhabitants of towns p per cent. have votes, and of country people q per cent. Also of voters r per cent. live in towns, and of non-voters s per cent. Find the proportion of the whole population who have votes; and show that p, q, r, s are connected by the one relation $100(qr-ps) = (p+s)\,qr-(q+r)\,ps$.

Solution by E. F. ELTON, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let V and N = total numbers of voters and non-voters; then

$$100-p : p = \text{town non-voters} : \text{town voters} = \frac{s}{100}\,\mathrm{N} : \frac{r}{100}\,\mathrm{V},$$

and

$$100-q : q = \text{country non-voters} : \text{country voters} = \frac{100-s}{100}\,\mathrm{N} : \frac{100-r}{100}\,\mathrm{V},$$

therefore $$\frac{100-p}{p}\cdot\frac{r}{s} = \frac{\mathrm{N}}{\mathrm{V}} = \frac{100-q}{q}\cdot\frac{100-r}{100-s}.$$

Hence $$100\,(qr-ps) = (p+s)\,qr-(q+r)\,ps,$$

and $$\frac{ps}{100r-pr+ps} = \frac{\mathrm{V}}{\mathrm{N}+\mathrm{V}} = \text{number of voters : whole population.}$$

9403. (RUSTICUS.)—Baby TOM of baby HUGH the nephew is and uncle too. In how many ways can this be true?

Solutions by (1) Professor MACFARLANE, LL.D.; (2) D. BIDDLE.

1. Using the method of the analysis of relationships described by me in Vol. XXXVI., p. 78, let T denote Tom, H Hugh, m male, f female, c child, c^{-1} parent; then the data are

$$\mathrm{T} = mc\,c\,c^{-1}m\mathrm{H},\quad \mathrm{T} = mc\,c^{-1}c^{-1}m\mathrm{H}\;\ldots\ldots\ldots\ldots\ldots(1, 2).$$

By combining the two, we obtain $\mathrm{H} = mc\,c\,c^{-1}mc\,c\,c^{-1}m\mathrm{H}$ ....(3).
Now, if the sex symbol before the third and fifth symbols of relationship are the same, the equation reduces to

$$\mathrm{H} = mc\,c\,c\,c^{-1}\,m\mathrm{H},\quad i.e.,\ c^{-1}m\mathrm{H} = c\,c\,c^{-1}m\mathrm{H}\;\ldots\ldots\ldots\ldots(4).$$

Now the sex of c^{-1} on each side cannot be the same, for then cc would be 1; that is, a person could be identical with his or her own grandchild. Nor can the sex be different, for then one parent would be the grandchild of the other parent, which is against the law of marriage. Hence the sex symbol in the third and fifth places cannot be the same.

Nor can the sex symbol in the second and sixth places be the same, for (3) is equivalent to $c^{-1}m\mathrm{H} = c\,c^{-1}\,mc\,c\,c^{-1}\,m\mathrm{H}$........................(5);
and, under the hypothesis, the sex of the parent of H is the same; then

$$c\,c^{-1}\,mc\,c = 1,\quad\text{or}\quad c^{-1}\,mc\,c = c^{-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Now the sex of the first and third symbols cannot be the same, for then $c = c^{-1}$, *i.e.*, the child of a person could be the parent of the person; nor can the symbols be different, for then the consort of the child of a person could be a parent of the person, which is against the law of marriage. The only cases left are

$$mc\,mc\,mc^{-1}\,mc\,fc\,fc^{-1}\,m = 1,\qquad mc\,fc\,fc^{-1}\,mc\,mc\,mc^{-1}\,m = 1\;\ldots(7, 8),$$
$$mc\,mc\,fc^{-1}\,mc\,fc\,mc^{-1}\,m = 1,\qquad mc\,fc\,mc^{-1}\,mc\,mc\,fc^{-1}\,m = 1\ldots(9, 10).$$

Now (7) means that Hugh's father and Tom's mother have married each the appropriate parent of the other; (8) that Hugh's mother and Tom's father have married each the appropriate parent of the other; (9) that the fathers of the two babies have married each the mother of the other; and (10) that the mothers of the babies have married each the father of

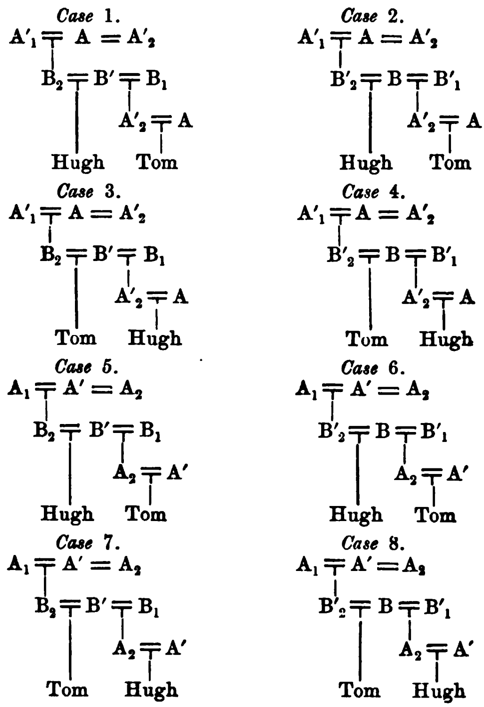
the other. Hence there are four, and only four, possible ways in which the phenomenon may be true.

2. *Otherwise* :—In relation to an uncle, the species "nephew" comprises the following varieties: (1) a brother's son, (2) a sister's son, (3) a half-brother's son, (4) a half-sister's son, (5) a wife's brother's son, (6) a wife's sister's son, (7) a wife's half-brother's son, (8) a wife's half-sister's son. This list comprehends all the varieties.

Of these eight varieties, the four last are excluded from our present consideration by the epithet "baby," attached to both individuals, which puts the existence of a wife of either out of the question.

Again, although we may suppose a grandfather to marry a granddaughter, and that she and her mother bears sons about the same time, one named Tom, and the other Hugh; and, although we may suppose, what is still more preposterous, that a grandmother marries her grandson, and bears a son about the same time his mother does; such marriages are contrary to law, so that we must exclude also varieties (1) and (2), and restrict our attention to (3) and (4).

Let A, B represent husbands, and A′, B′ their respective wives, whilst figures at the foot signify first or second. We then obtain the following eight cases :—



It is quite possible, in these cases, that no disparity of age shall exist between A and A'_2, or between A' and A_2.

Note on a Rectangular Hyperbola. By R. Tucker, M.A.

The equations to the hyperbola with respect to which the triangle of reference ABC is self-conjugate, and to a tangent (or polar) thereto, are

$$\phi \equiv \alpha^2 (b^2-c^2) + \beta^2 (c^2-a^2) + \gamma^2 (a^2-b^2) = 0 \quad\ldots\ldots\ldots\ldots\ldots (1),$$

$$\alpha\alpha' (b^2-c^2) + \beta\beta' (c^2-a^2) + \gamma\gamma' (a^2-b^2) = 0 \quad\ldots\ldots\ldots\ldots\ldots (2).$$

From (2) we see that the polar of either of a pair of inverse points, *i.e.*, foci of an in-conic, passes through the other point. The same property holds for the related points

$$\frac{\alpha_1\alpha_2}{a^2} = \frac{\beta_1\beta_2}{b^2} = \frac{\gamma_1\gamma_2}{c^2}, \quad \text{and} \quad \frac{\alpha_1\alpha_2}{b^2+c^2} = \frac{\beta_1\beta_2}{c^2+a^2} = \frac{\gamma_1\gamma_2}{a^2+b^2} \quad\ldots\ldots (3).$$

From (1) we see that the curve passes through the circumcentre (O)*, the Symmedian point (K), and the in- and ex-centres. Its centre, which, of course, lies upon the circumcircle, is

$$\alpha (b^2-c^2)/a = \beta (c^2-a^2)/b = \gamma (a^2-b^2)/c \quad\ldots\ldots\ldots\ldots\ldots (4),$$

the "inverse" of which is the "isotomic" of the Steiner point

$$[a\alpha (b^2-c^2) = \ldots = \ldots].$$

The tangent at the in-centre is $\alpha (b^2-c^2) + \ldots + \ldots = 0$, a line which passes through (a^2, b^2, c^2), $(b^2+c^2, c^2+a^2, a^2+b^2)$,

$(\cot A, \cot B, \cot C)$, and $(1/b+c, 1/c+a, 1/a+b)$ ...... (5).

The tangent at K is $a\alpha (b^2-c^2) + \ldots + \ldots = 0$, which passes through the centroid $(1/a(b+c), \ldots)$, $(\cot A/a, \cot B/b, \cot C/c)$, the last point in (5) and other points through which $K\pi$ passes......... (6). (See Note on "Symmedian-point Axis, &c." Q. J., Vol. xx., No. 78.)

The tangent at O is $\alpha \cos A (b^2-c^2) + \ldots + \ldots = 0$ .... (7), which passes through the orthocentre and centroid.

The polar of the centroid is $\alpha (b^2-c^2)/a + \ldots + \ldots = 0$, which is the circum-Brocardal axis.

The polar of the Steiner point is $\alpha/a + \beta/b + \gamma/c = 0$, that is, the axis of perspective of ABC and of the triangle formed by tangents at A, B, C to the circumcircle. The "isotomic" of O is the point $(1/a^2 \cos A, \ldots, \ldots)$, *i.e.*, the point P of my paper on "Isoscelians" (cf. *Proc. L. Math. Soc.*, Vol. xix., No. 315), hence the inverse of P is $(a^2 \cos A, \ldots, \ldots)$, the polar of which is $a^2 \alpha \cos A (b^2-c^2) + \ldots + \ldots = 0$ ....(8), *i.e.*, the line $P\pi$ of the above cited Note (Q. J., Vol. xx.).

The polar of the centre of the Brocard ellipse is

$$a\alpha (b^4-c^4) + \ldots + \ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

[* This follows also from the fact that O is the orthocentre of the ex-centric triangle.]

which passes through the centroid and the inverse of Kiepert's point (a^3, b^3, c^3).

Similarly we can obtain results from the polars of the two Brocard points.

The asymptotes are given by $P\phi + b^2-c^2 . c^2-a^2 . a^2-b^2 (a\alpha + \ldots + \ldots)^2 = 0$.

where $$P = a^2b^2c^2 (1-8\cos A \cos B \cos C).$$

9479. (A. Kahn, M.A.)—Solve the equations $xyz = 24$,
$x(y-z)^2 + y(z-x)^2 + z(x-y)^2 = 18, \quad x^2(y-z) + y^2(z-x) + z^2(x-y) = -2$.

Solution by R. Knowles, B.A.; H. L. Orchard, M.A., B.Sc.; *and others.*

Let $y = mx$, $z = nx$; then

$$x^3 = 80/(m+m^2n+n^2) = 82/(n+mn^2+m^2) = 24/mn.$$

Eliminating m from these equations, there results

$$12n^3-41n^2+40n-12 = 0, \text{ or } (n-2)(4n-3)(3n-2) = 0,$$

whence $n = 2, \frac{3}{4}, \frac{2}{3}$, and $m = \frac{3}{2}, \frac{4}{3}, \frac{1}{2}$; and, since $x^3 = 24/mn$, therefore, when $m = \frac{3}{2}$, $n = 2$, $\quad x = 2, \; y = 3, \; z = 4$.

[By inspection we have $x = 2, y = 3, z = 4$; and the symmetry of the equations shows that congruent values are

$$x = 3, \; y = 4, \; z = 2; \quad x = 4, \; y = 2, \; z = 3.]$$

9183. (A. R. Johnson, M.A.)—Investigate the induced magnetisation of an ellipsoidal shell composed of any number of strata bounded by confocal surfaces.

Solution by the Proposer.

Let there be m strata bounded by confocal surfaces which may be conveniently indicated by the suffixes 0, 1, 2, ... m, counting outwards.

Let μ_n, V_n be the magnetic inductive capacity and the total potential in the stratum between the $(n-1)^{\text{th}}$ and n^{th} surfaces. Then, if the inducing potential be ES, the proper assumption is $V_n = (A_nE + B_nF)S$. The conditions at the n^{th} surface give

$$A_{n+1}E_n + B_{n+1}F_n = A_nE_n + B_nF_n,$$

$$\mu_{n+1}(A_{n+1}\dot{E}_n + B_{n+1}\dot{F}_n) = \mu_n(A_n\dot{E}_n + B_n\dot{F}_n);$$

whence $$a_nA_{n+1} = -g_nA_n - e_nB_n, \quad a_nB_{n+1} = b_nA_n + f_nB_n \quad \ldots\ldots\ldots\ldots(1),$$

where $$\left.\begin{aligned} a_n &= \frac{E_n}{F_n} - \frac{\dot{E}_n}{\dot{F}_n}, \quad b_n = \left(\frac{\mu_n}{\mu_{n+1}} - 1\right)\frac{E_n\dot{F}_n}{F_n\dot{F}_n}, \quad e_n = \frac{\mu_n}{\mu_{n+1}} - 1 \\ f_n &= \frac{\mu_n}{\mu_{n+1}}\frac{E_n}{F_n} - \frac{\dot{E}_n}{\dot{F}_n}, \quad g_n = \frac{\mu_n}{\mu_{n+1}}\frac{\dot{E}_n}{\dot{F}_n} - \frac{E_n}{F_n} \end{aligned}\right\} \ldots\ldots(2).$$

From (1) $$\frac{a_{n+1}}{c_{n+1}}\mathrm{A}_{n+2}-\left(\frac{f_n}{c_n}-\frac{g_{n+1}}{c_{n+1}}\right)\mathrm{A}_{n+1}+\left(\frac{b_n}{a_n}-\frac{f_n g_n}{a_n c_n}\right)\mathrm{A}_n=0 \quad\text{......(3).}$$

The internal potential V_0 cannot contain F. Therefore $\mathrm{B}_0=0$, and therefore, from (1), $a_0\,\mathrm{A}_1=-g_0\,\mathrm{A}_0$.

From (3) and from the relation just obtained, there results, after some reductions,

$$^{*}\frac{\mathrm{A}_{n+1}}{\mathrm{A}_n}=\left\{\frac{\mu_n(\mu_{n+1}-\mu_n)}{\mu_{n+1}(\mu_n-\mu_{n-1})}\left[\frac{\mu_{n+1}-\mu_{n-1}}{\mu_{n+1}-\mu_n}-\frac{\mu_n(\beta_n-\beta_{n-1})-\mu_{n+1}(\beta'_n-\beta'_{n-1})}{a_n}\right]\right.$$

$$\left.-\frac{\mu_{n+1}-\mu_n}{\mu_n-\mu_{n-1}}\,\frac{a_{n-1}}{a_n}\right\}\Big/\left\{\frac{\mu'_{n-1}}{\mu_n}-\frac{(\mu_n-\mu_{n-1})}{(\mu_{n-1}-\mu_{n-2})}\times\right.$$

$$\left.\left[\frac{\mu_n-\mu_{n-2}}{\mu_n-\mu_{n-1}}-\frac{\mu_{n-1}(\beta_{n-1}-\beta_{n-2})-\mu_n(\beta'_{n-1}-\beta'_{n-2})}{a_{n-1}}\right]-\frac{\mu_n-\mu_{n-1}}{\mu_{n-1}-\mu_{n-2}}\,\frac{a_{n-2}}{a_{n-1}}\right\}\Big/$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$\left\{\frac{\mu_1(\mu_2-\mu_1)}{\mu_2(\mu_1-\mu_0)}\left[\frac{\mu_2-\mu_0}{\mu_2-\mu_1}-\frac{\mu_1(\beta_1-\beta_0)-\mu_2(\beta_1'-\beta_0')}{a_1}\right]-\frac{\mu_2-\mu_1}{\mu_1-\mu_0}\,\frac{a_0}{a_1}\right\}\Big/$$

$$\left\{\frac{\mu_1\beta_0'-\mu_0\beta_0}{a_0}\right\}\text{..............................(4),}$$

where $$\beta_n=\frac{\mathrm{E}_n}{\mathrm{F}_n},\quad \beta'_n=\frac{\dot{\mathrm{E}}_n}{\dot{\mathrm{F}}};\quad a_n=\mu_n(\beta_n-\beta'_n).$$

Hence $\mathrm{A}_{n+1}/\mathrm{A}_0$ = product of continued fraction (4) and those derived from it by writing $n-1$, $n-2$, etc., instead of n; the last being $(\mu_1\beta_0-\mu_0\beta_0')/a_0$. That is to say, $\mathrm{A}_{n+1}/\mathrm{A}_0=\mathrm{N}_{n+1}$, where N_{n+1} is the numerator of the continued fraction (4) expressed as a proper fraction by the method of convergents.

Now the potential due to the induced magnetisation can have no term in E in external space; *i.e.*, $\mathrm{A}_{m+1}=1$. Therefore $\mathrm{A}_0=1/\mathrm{N}_{m+1}$, and

$$\mathrm{A}_n=\mathrm{N}_n/\mathrm{N}_{m+1}\text{................................(5).}$$

We may in the same way find B_n from the equation

$$\frac{a_{n+1}}{b_{n+1}}\mathrm{B}_{n+2}-\left(\frac{f_n}{b_n}-\frac{g_{n+1}}{b_{n+1}}\right)\mathrm{B}_{n+1}+\left(\frac{c_n}{a_n}-\frac{f_n g_n}{a_n b_n}\right)\mathrm{B}_n=0,$$

or proceed as follows.

From (1) when $n \not> m$, $$\mathrm{B}_n=\frac{a_n\,\mathrm{N}_{n+1}+g_n\,\mathrm{N}_n}{c_n\,\mathrm{N}_{m+1}}\text{ ....(6),}$$

and $$\mathrm{B}_{m+1}=\frac{b_m}{a_m}\mathrm{A}_m+\frac{f_m}{a^m}\mathrm{B}_m,$$

so that $$(\mu_{m+1}-\mu_m)\,\mathrm{B}_{m+1}=\frac{\mathrm{N}_m}{\mathrm{N}_{m+1}}a_m+\mu_{m+1}\beta'_m-\mu_m\beta_m\text{.............(7).}$$

[* When the fraction is written in the customary way, the last term within each each pair of parentheses is to be placed over all that follows it.]

In the case of a composite spherical shell, $E = r^i$, $F = r^{-i-1}$, and

$$\beta_r = r_n^{2i+1},\quad \beta'_n = -\frac{i}{i+1} r_n^{2i+1},\quad a_n = \frac{2i+1}{i+1} r_n^{2i+1},$$

so that (4) becomes

$$\left\{\frac{\mu_n(\mu_{n+1}-\mu_n)}{\mu_{n+1}(\mu_n-\mu_{n-1})}\left[\frac{\mu_{n+1}-\mu_{n-1}}{\mu_{n+1}-\mu_n} - \frac{\{i\mu_{n+1}+(i+1)\mu_n\}\{1-(r_{n-1}/r_n)^{2i+1}\}}{(2i+1)\mu_n}\right]\right.$$
$$\left. - \frac{\mu_{n-1}(\mu_{n+1}-\mu_n)}{\mu_n(\mu_n-\mu_{n-1})}\left(\frac{r_{n-1}}{r_n}\right)^{2i+1}\right\} \Big/ \text{etc.} \quad \Big/ -\frac{i\mu_1+(i+1)\mu_0}{(2i+1)\mu_0}.$$

From the complete potential subtract the inducing potential, and the result is the potential of the induced magnetisation.

8853. (A. Russell, B.A.)—Prove that

$$v = \int_0^\infty \int^\infty \int f\left(t - \frac{x^2}{4a^2n^2},\ t - \frac{y^2}{4b^2p^2},\ t - \frac{z^2}{4c^2q^2}\right) e^{-(n^2+p^2+q^2)}\, dn\, dp\, dq$$

is a solution of the differential equation

$$\frac{dv}{dt} = a^2\frac{d^2v}{dx^2} + b^2\frac{d^2v}{dy^2} + c^2\frac{d^2v}{dz^2}.$$

Solution by the Proposer.

Since

$$v = \int_0^\infty \int_0^\infty \int_0^\infty f\left(t - \frac{x^2}{4a^2n^2},\ t - \frac{y^2}{4b^2p^2},\ t - \frac{z^2}{4c^2q^2}\right) e^{-(n^2+p^2+q^2)}\, dn\, dp\, dq,$$

therefore $$\frac{dv}{dt} = 3\int_0^\infty \int_0^\infty \int_0^\infty f'(\ \dots\ )\, e^{-(n^2+p^2+q^2)}\, dn\, dp\, dq,$$

also

$$\frac{dv}{dx} = -\frac{1}{a}\int_0^\infty \int_0^\infty \int_0^\infty f\left(t-n^2,\ t - \frac{y^2}{4b^2p^2},\ t - \frac{z^2}{4c^2q^2}\right) e^{-x^2/(4a^2n^2)-p^2-q^2}\, dn\, dp\, dq;$$

therefore

$$a^2\frac{d^2v}{dx^2} = \int_0^\infty \int_0^\infty \int_0^\infty f'\left(t - \frac{x^2}{4a^2n^2},\ t - \frac{y^2}{4b^2p^2},\ t - \frac{z^2}{4c^2q^2}\right) e^{-(n^2+p^2+q^2)}\, dn\, dp\, dq.$$

Thus we see v satisfies the given differential equation. This solution of the equation of the motion of heat in an eolotropic solid is suitable for the case of a time-periodic source.

9524. (Rev. J. J. Milne, M.A.)—If y_1, y_2, y_3 are the ordinates of three points P, Q, R on the parabola $y^2 = 4ax$, such that the circle on PQ as diameter touches the parabola at R, prove that

$$y_1 + y_2 = 2y_3,\quad y_1 \sim y_2 = 8a.$$

Solution by A. E. THOMAS, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

(1) is derived from the well-known theorem, "The algebraic sum of the ordinates of the points of intersection of a circle and parabola is zero," by supposing two of the points to coincide.

(2) The points where the circle on m_1, m_2, as diameter meets the parabola again are given by

$$(am^2 - am_1^2)(am^2 - am_2^2) + (2am_1 - 2am)(2am_2 - 2am) = 0,$$

or, discarding the factor $a^2(m-m_1)(m-m_2)$,

$$m^2 + (m_1+m_2)m + m_1 m_2 + 4 = 0,$$

which has equal roots if $m_1 \sim m_2 = 4$, *i.e.* if $y_1 \sim y_2 = 8a$.

9267. (Professor HANUMANTA RAU, M.A.)—Given the base and the vertical angle of a triangle, prove that the envelope of the nine-points circle is itself a circle.

Solution by R. F. DAVIS, M.A.; D. O. S. DAVIES, M.A.; *and others.*

Since the circum-circle is fixed, the radius of the nine-points circle is also fixed. The nine-points circle therefore touches, at the other extremity of its diameter through the mid-point of the base, a circle having this latter point for centre and radius equal to the circum-radius.

9314. (Professor BENI MADHAV SARKAR, B.A.)—Solve the equation

$$x + yz = a = 384, \quad y + zx = b = 237, \quad z + xy = c = 192.$$

Solution by R. F. DAVIS, M.A.; D. WATSON, M.A.; *and others.*

Since $x = a - yz = (b-y)/z = (c-z)/y$, we find $y = (az-b)/(z^2-1)$ and $x = (bz-a)/(z^2-1)$; hence $(z-c)(z^2-1)^2 + (az-b)(bz-a) = 0$. There are, therefore, five solutions as each value of z determines a single value for both x and y. In the given example, $x = 10$, $y = 17$, $z = 22$ by trial. The equation determining z is the subjoined quintic,

$$z^5 - 192z^4 - 2z^3 + 91392z^2 - 203624z + 90816 = 0,$$

which is reducible to the following quartic by rejecting the root 22,

$$z^4 - 170z^3 - 3742z^2 + 9068z - 4128 = 0, \quad \text{or}$$

$$(z^2 - 85z + 40{\cdot}728)^2 = 11048{\cdot}46z^2 - 15992z + 5786{\cdot}77 = (105{\cdot}11z - 76{\cdot}07)^2 \text{ \&c.},$$

whence $z = 189{\cdot}6,\ 1{\cdot}62,\ {\cdot}63,\ -21{\cdot}9$ roughly; so that $(-11{\cdot}7,\ -18{\cdot}1,\ -21{\cdot}9)$, $(-{\cdot}8,\ 238,\ 1{\cdot}62)$, $(1{\cdot}24,\ 2{\cdot}02,\ 189{\cdot}6)$, $(380,\ {\cdot}5,\ {\cdot}63)$ are approximately the other four solutions.

9092. (A. E. JOLLIFFE, M.A.)—Prove that

$$\frac{(2n)!}{n!\,n!}-\frac{(2n-1)!}{1!\,(n-1)!\,(n-1)!}+\frac{(2n-2)!}{2!\,(n-2)!\,(n-2)!}-\dots \text{ to } (n+1) \text{ terms} = 1.$$

Solution by A. E. THOMAS, M.A.; Prof. MATZ; *and others.*

$$x^r(1+x)^n = (1+x)^n[(1+x)-1]^r$$
$$= (1+x)^{n+r}-r(1+x)^{n+r-1}+\frac{r(r-1)}{1.2}(1+x)^{n+r-2}-\dots$$

Equating coefficients of x^r on both sides we get

$$1 = \frac{(n+r)!}{r!\,n!}-r\frac{(n+r-1)!}{r!\,(n-1)!}+\frac{r(r-1)}{2!}\frac{(n+r-2)!}{r!\,(n-2)!}\dots \text{ to } (r+1) \text{ terms}$$
$$= \frac{(n+r)!}{r!\,n!}-\frac{(n+r-1)!}{1!\,(r-1)!\,(n-1)!}+\frac{(n+r-2)!}{2!\,(r-2)!\,(n-2)!}\dots \text{ to } (r+1) \text{ terms}.$$

If $r = n$, we have

$$1 = \frac{(2n)!}{n!\,n!}-\frac{(2n-1)!}{1!\,(n-1)!\,(n-1)!}+\frac{(2n-2)!}{2!\,(n-2)!\,(n-2)!}\dots \text{ to } (n+1) \text{ terms}.$$

8782. (A. RUSSELL, B.A.)—Prove that, if

$$a^3(b+c)+b^3(c+a)+c^3(a+b) = 2abc(a+b+c), \text{ then}$$

$$(1)\quad \left(\frac{b^2+c^2}{a}-2a\right)\Big/\left(\frac{2bc}{b+c}-a\right) = \left(\frac{c^2+a^2}{b}-2b\right)\Big/\left(\frac{2ca}{c+a}-b\right)$$
$$= \left(\frac{a^2+b^2}{c}-2c\right)\Big/\left(\frac{2ab}{a+b}-c\right);$$

$$(2)\quad (a^{8n}-b^{4n}c^{4n})(a^2-bc)(b+c)^2(b^2+c^2)(b^4+c^4)\dots(b^{2n}+c^{2n})+\dots+\dots = 0;$$

$$(3)\quad (b^2-c^2)\left(a-\frac{2bc}{b+c}\right)^3\left\{3a^2+a(b+c)+bc\right\}$$
$$+(c^2-a^2)\left(b-\frac{2ca}{c+a}\right)^3\left\{3b^2+b(c+a)+ca\right\}$$
$$+(a^2-b^2)\left(c-\frac{2ab}{a+b}\right)^3\left\{3c^2+c(a+b)+ab\right\} = 0.$$

Solution by the PROPOSER; Professor SARKAR, M.A.; *and others.*

1, 2. The given relation may be written

$(a^2-bc)(b+c)\,a+\dots+\dots = 0$, and since $(a^2-bc)(b+c)+\dots+\dots = 0$,

therefore $$(a^2-bc)\frac{b+c}{b-c} = (b^2-ca)\frac{c+a}{c-a} = (c^2-ab)\frac{a+b}{a-b} \quad\dots\dots\dots\dots(a).$$

Similarly $$\frac{ab+ac-2bc}{b^2-c^2} = \frac{bc+ba-2ca}{c^2-a^2} = \frac{ca+cb-2ab}{a^2-b^2} \quad\dots\dots\dots\dots(b),$$

and $$\frac{c^2+b^2-2a^2}{a(b-c)} = \frac{c^2+a^2-2b^2}{b(c-a)} = \frac{a^2+b^2-2c^2}{c(a-b)} \quad\dots\dots\dots\dots(c).$$

From (a) we deduce (2), and from (b) and (c) (1) follows.

3. The given relation may also be written

$$a-\frac{2bc}{b+c}+b-\frac{2ca}{c+a}+c-\frac{2ab}{a+b}=0;$$

and putting $x=a-\dfrac{2bc}{b+c}$, $y=$ &c., $z=$ &c., $x+y+z=0$,

and therefore $\quad x^3(y-z)+y^3(z-x)+z^3(x-y)=0.$

Substituting for x, y, &c. their values, after a little reduction, the required result follows.

9416. (J. O'BYRNE CROKE, M.A. Suggested by Question 9360.)—The sides of a polyhedron are of areas inversely as the perpendiculars on them from a point O, and OO′ meets them in $P_1, P_2, P_3 \dots P_n$, respectively; prove that $\dfrac{O'P_1}{OP_1}+\dfrac{O'P_2}{OP_2}+\dfrac{O'P_3}{OP_3}+\dots+\dfrac{O'P_n}{OP_n}=n.$

Solution by Professors CURTIS, M.A.; BEYENS; *and others.*

The volumes of the pyramids which are subtended by the sides of the polyhedron at O are equal each to $n^{-1}V$, where $V =$ vol. of polyhedron; hence, calling the vols. subtended by these sides at O′, v_1, v_2, v_3, &c., we have

$$\frac{O'P_1}{OP_1}+\frac{O'P_2}{OP_2}+\dots\text{\&c.}=\frac{v_1+v_2+\text{\&c.}}{n^{-1}V}=\frac{V}{n^{-1}V}=n.$$

9256. (E. VIGARIE.)—Dans un triangle ABC si (α) est le pied sur BC de la symédiane issue du sommet A, et si (α') est le point conjugué harmonique de (α); démontrer que $A\alpha'$ est égale au rayon du cercle d'Apollonius correspondant à BC.

Solution by Professor IGNACIO BEYENS.

De la propriété de la symédiane bien connue $\dfrac{\alpha B}{\alpha C}=\dfrac{c^2}{b^2}$ on déduit

$$\frac{\alpha' B}{\alpha' C}=\frac{\alpha B}{\alpha C}=\frac{c^2}{b^2},\text{ d'où } \alpha' B=\frac{ac^2}{c^2-b^2},\quad \alpha' C=\frac{ab^2}{c^2-b^2};$$

mais si (m) et (m') sont les pieds des bissectrices qui partent du sommet A, on aura $mC=\dfrac{ab}{b+c}$ et $Cm'=\dfrac{ab}{c-b}$, d'où le rayon du cercle d'Apollonius

sera $$\frac{mm'}{2}=\frac{mC+Cm'}{2}=\frac{abc}{c^2-b^2}=\rho \quad \dots\dots\dots\dots\dots\dots\dots(1).$$

Mais, par le théorème de Stewart, dans le triangle $AB\alpha'$ nous aurons

$$A\alpha'^2 \,.\, BC+AB^2 \,.\, C\alpha' = B\alpha'\,(AC^2+BC \,.\, C\alpha'),$$

et remplaçant Ca', Ba', AB, AC, BC par leurs valeurs on aura

$$Aa'^2 . a = \frac{a^3 c^{\cdot} b^2}{(c^2 - b^2)^2}, \text{ d'où } Aa' = \frac{abc}{c^2 - b^2} = \rho,$$

rayon du cercle d'Apollonius d'après (1).

9412. (A. R. Johnson, M.A.)—Show that, if 1, 2, 3, 4, 5, 6 be six points on a conic, then $0 = \Sigma\,(023)\,(031)\,(012)\,(456)$, Σ denoting summation with respect to all terms obtained from the one presented by cyclic interchanges; O denoting any point in the plane of the conic, and (456), etc. the areas of the triangles 456, etc., described in the order named.

Solution by Professor Curtis, M.A.; G. G. Storr, M.A.; *and others.*

If the six points (x_1, y_1), (x_2, y_2), &c., all lie on the conic

$$(a, b, c, f, g, h \between x, y, 1) = 0,$$

we must have six linear equations in a, b, c, f, g, h, the condition for whose being simultaneous is the determinant

$$\begin{vmatrix} x_1^2, & x_1y_1, & y_1^2, & x_1, & y_1, & 1 \\ x_2^2, & x_2y_2, & y_2^2, & x_2, & y_2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_6^2, & x_6y_6, & y_6^2, & x_6, & y_6, & 1 \end{vmatrix} = 0, \text{ or } \Sigma \begin{vmatrix} x_1^2, & x_1y_1, & y_1^2 \\ x_2^2, & x_2y_2, & y_2^2 \\ x_3^2, & x_3y_3, & y_3^2 \end{vmatrix} \times \begin{vmatrix} x_4, & y_4, & 1 \\ x_5, & y_5, & 1 \\ x_6, & y_6, & 1 \end{vmatrix} = 0.$$

Now, the first of the two determinants here written down is equivalent to

$$-(x_2y_3 - x_3y_2)(x_3y_1 - x_1y_3)(x_1y_2 - x_2y_1),$$

or to eight times the product of the triangles (023), (031), (012), and the other determinant is double the triangle (456). Hence the theorem stated.

If we make O coincide with the point (x_6, y_6), we see that the condition that six points may lie on one conic may be written

$$\Sigma\,(023)\,(031)\,(012) \times (045) = 0,$$

there being ten such products to be taken each with its proper sign. Of course, each when developed would express the property of Pascal's Hexagram.

8766. (S. Tebay, B.A.)—If AX, BY, CZ be opposite dihedral angles of a tetrahedron, show how to construct the solid in order that

$$\begin{aligned} &\left\{\tan \tfrac{1}{2}(B-Y) - \tan \tfrac{1}{2}(C-Z)\right\} \tan \tfrac{1}{2}(A+X) \\ &+ \left\{\tan \tfrac{1}{2}(C-Z) - \tan \tfrac{1}{2}(A-X)\right\} \tan \tfrac{1}{2}(B+Y) \\ &+ \left\{\tan \tfrac{1}{2}(A-X) - \tan \tfrac{1}{2}(B-Y)\right\} \tan \tfrac{1}{2}(C+Z) = 0. \end{aligned}$$

Solution by the PROPOSER.

Let a, b, c be conterminous edges; x, y, z the opposites, and Δ_1, Δ_2, Δ_3, Δ_4 the areas of the faces bcx, cay, abz, xyz; then

$$\text{volume} = \tfrac{2}{3}\,\frac{\Delta_2\Delta_3}{a}\sin A = \tfrac{2}{3}\,\frac{\Delta_1\Delta_4}{x}\sin X;$$

therefore $$\frac{\Delta_1\Delta_2\Delta_3}{\Delta_4} = \frac{a\sin X}{x\sin A}\,\Delta_1^2 = \frac{b\sin Y}{y\sin B}\,\Delta_2^2 = \frac{C\sin Z}{z\sin C}\,\Delta_3^2.$$

If $\dfrac{\sin A}{\sin X} = \dfrac{\sin B}{\sin Y} = \dfrac{\sin C}{\sin Z}$, we have

$$\tan\tfrac{1}{2}(A+X) = M\tan\tfrac{1}{2}(A-X),\quad \tan\tfrac{1}{2}(B+Y) = M\tan\tfrac{1}{2}(B-Y),$$
$$\tan\tfrac{1}{2}(C+Z) = M\tan\tfrac{1}{2}(C-Z);$$

M being a common factor. Whence the proposed relation. We also have

$$\frac{a}{x}\,\Delta_1^2 = \frac{b}{y}\,\Delta_2^2 = \frac{c}{z}\,\Delta_3^2.$$

These conditions furnish the required construction.

9354. (Professor MAHENDRA NATH RAY, M.A., LL.B.)—A pencil of four rays radiates from the middle point of the base of a triangle, and is terminated by the sides. If the segments of the rays measured from the origin be x_1, y_1, x_2, y_2, x_3, y_3, and x_4, y_4, show that the identical relation connecting these lengths is

$$\begin{vmatrix} x_1^{-2}, & x_2^{-2}, & x_3^{-2}, & x_4^{-2} \\ y_1^{-2}, & y_2^{-2}, & y_3^{-2}, & y_4^{-2} \\ (x_1y_1)^{-1}, & (x_2y_2)^{-1}, & (x_3y_3)^{-1}, & (x_4y_4)^{-1} \\ 1, & 1, & 1, & 1 \end{vmatrix} = 0.$$

Solution by J. O'BYRNE CROKE, M.A.

Obviously the relation which is to be established must be independent of the position of the origin. Let ω = vertical angle of the Δ; and let ϵ, ϵ' be the parts into which it is divided by a line of length k drawn to the origin of rays. Then, we easily find that

$$1/k^2\sin^2\omega = x_1^{-2}\sin^2\epsilon + y_1^{-2}\sin^2\epsilon' - 2\,(x_1y_1)^{-1}\sin\epsilon\sin\epsilon'\cos\omega;$$

which may be written

$$\kappa_1 x_1^{-2} + \kappa_2 y_1^{-2} + \kappa_3 (x_1y_1)^{-1} + \kappa_4 = 0.$$

And, eliminating the constants between this and the three other similar equations, we have
$$\begin{vmatrix} x_1^{-2}, & y_1^{-2}, & (x_1y_1)^{-1}, & 1 \\ x^{-2}, & y^{-2}, & (x_2y_2)^{-1}, & 1 \\ x_3^{-2}, & y_3^{-2}, & (x_3y_3)^{-1}, & 1 \\ x_4^{-2}, & y_4^{-2}, & (x_4y_4)^{-1}, & 1 \end{vmatrix} = 0;$$

a relation identical with that in the Question.

APPENDIX I.

SOLUTIONS OF SOME UNSOLVED QUESTIONS.

BY W. J. CURRAN SHARP, M.A.

2144. (Professor WOLSTENHOLME, Sc.D.)—If from the highest point of a sphere an infinite number of chords be drawn to points uniformly distributed over the surface, and heavy particles be let fall down these chords simultaneously, their centre of inertia will descend with acceleration $\frac{1}{2}g$.

Solution.

Let A be the highest point, AB the vertical diameter of the sphere, and let P be a point such that BAP $= \theta$; the acceleration on a particle falling down AP is $g\cos\theta$, and its depth at time t is $\frac{1}{2}g\cos^2\theta \,.\, t^2$.

From symmetry it is evident that the centre of inertia lies in AB. Now the area of a belt bounded by horizontal small circles such that the lines to them from A makes angles θ and $\theta + \delta\theta = 2\pi r \sin 2\theta \times 2r\,\delta\theta$, and the depth of the centre of inertia is therefore

$$= \frac{\int_0^{\frac{1}{2}\pi} \frac{1}{2}g\cos^2\theta\, t^2 \times 4\pi r^2 \sin 2\theta\, d\theta}{\int_0^{\frac{1}{2}\pi} 4\pi r^2 \sin 2\theta\, d\theta} = \tfrac{1}{2}gt^2\, \frac{\int_0^{\frac{1}{2}\pi} \cos^3\theta \sin\theta\, d\theta}{\int_0^{\frac{1}{2}\pi} \cos\theta\sin\theta\, d\theta} = \tfrac{1}{4}gt^2,$$

and the centre of inertia descends AB, as a particle would which started from rest at A, under the action of a uniform acceleration $\frac{1}{2}g$.

I have, of course, assumed that the particles were equal, as the data would be insufficient without some rule as to their mass, and this simple supposition gives the correct result. All question might be avoided by inserting the word *equal*—thus, "and equal heavy particles be let fall."

The result may be confirmed by the fact that all the particles reach the sphere at the same time, viz., that in which the one which falls along AB reaches B, and that then the centre of inertia is at the centre, as it should be.

2146. (Professor NASH, M.A.)—D, E, F are the points where the bisectors of the angles of the triangle ABC meet the opposite sides. If x, y, z are the perpendiculars drawn from A, B, C respectively to the

opposite sides of the triangle DEF; p_1, p_2, p_3 those drawn from A, B, C respectively to the opposite sides of ABC: prove that

$$\frac{p_1^2}{x^2}+\frac{p_2^2}{y^2}+\frac{p_3^2}{z^2}=11+8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Solution.

$$AE=\frac{bc}{a+c},\quad AF=\frac{bc}{a+b}\ (\text{Euclid, VI. 3});$$

therefore the triangle AFE $=\frac{1}{2}\dfrac{b^2c^2}{(a+c)(a+b)}\sin A=\dfrac{x\,.\,FE}{2}$,

and $\quad FE^2=b^2c^2\left\{\dfrac{1}{(a+c)^2}+\dfrac{1}{(a+b)^2}-\dfrac{2}{(a+c)(a+b)}\cos A\right\}$

$$=\frac{b^2c^2}{(a+c)^2(a+b)^2}\left\{2a^2+2ab+2ac+b^2+c^2-2(a^2+ab+ac+bc)\cos A\right\}$$

$$=\frac{b^2c^2}{(a+c)^2(a+b)^2}\left\{2a^2(1-\cos A)+2a(b-c\cos A)+2a(c-b\cos A)+a^2\right\}$$

$$=\frac{2a^2b^2c^2}{(a+c)^2(a+b)^2}\left\{\tfrac{3}{2}-\cos A+\cos B+\cos C\right\};$$

$$\therefore\quad \frac{a^2b^2c^2x^2}{2(a+c)^2(a+b)^2}\left\{\tfrac{3}{2}-\cos A+\cos B+\cos C\right\}=\frac{1}{4}\frac{b^4c^4}{(a+c)^2(a+b)^2}\sin^2 A;$$

therefore $\quad \frac{1}{2}(x^2a^2)\left\{\frac{3}{2}-\cos A+\cos B+\cos C\right\}=\frac{1}{4}b^2c^2\sin^2 A=\frac{1}{4}p_1^2a^2;$

therefore $\quad 3-2\cos A+2\cos B+2\cos C=p_1^2/x^2.$

Similarly, $\quad 3+2\cos A-2\cos B+2\cos C=p_2^2/y^2,$

$\quad 3+2\cos A+2\cos B-2\cos C=p_3^2/z^2;$

therefore
$$\frac{p_1^2}{x^2}+\frac{p_2^2}{y^2}+\frac{p_3^2}{z^2}=9+2(\cos A+\cos B+\cos C)$$
$$=11+8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

2173. (Professor WOLSTENHOLME, Sc.D.)—The quadric
$$ax^2+by^2+cz^2=1$$
is turned about its centre until it touches $a'x^2+b'y^2+c'z^2=1$ along a plane section. Find the equation to this plane section referred to the axes of either of the quadrics, and show that its area is
$$\pi(a+b+c-a'-b'-c')^{\frac{1}{2}}/(abc-a'b'c')^{\frac{1}{2}}.$$

Solution.

Let $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=1$ be the equation to the quadric, the original equation of which was $ax^2+by^2+cz^2=0$, after it has been turned about its centre. Then, by question,

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy\equiv a'x^2+b'y^2+c'z^2$$
$$+R(x\cos\alpha+y\cos\beta+z\cos\gamma)^2,$$

therefore $\quad A = a' + R\cos^2\alpha, \quad B = b' + R\cos^2\beta, \quad C = c' + R\cos^2\gamma,$

$\quad F = R\cos\beta\cos\gamma, \quad G = R\cos\gamma\cos\alpha, \quad H = R\cos\alpha\cos\beta.$

Now, by a Paper in the *Proc. Lond. Math. Soc.*, Vol. XIII., pp. 193—94,

$$A + B + C = a + b + c,$$

$$AB + BC + CA - F^2 - G^2 - H^2 = ab + bc + ca,$$

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = abc;$$

therefore $\quad a + b + c = a' + b' + c' + R, \quad$ because $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$

$$\begin{aligned} ab + bc + ca &= a'b' + b'c' + c'a' + R\left\{a'(\cos^2\beta + \cos^2\gamma)\right. \\ &\qquad \left. + b'(\cos^2\gamma + \cos^2\alpha) + c'(\cos^2\alpha + \cos^2\beta)\right\} \\ &= a'b' + b'c' + c'a' + R\left\{a' + b' + c' - a'\cos^2\alpha - b'\cos^2\beta - c'\cos^2\gamma\right\}, \\ abc &= a'b'c' + R\left\{b'c'\cos^2\alpha + c'a'\cos^2\beta + a'b'\cos^2\gamma\right\}; \end{aligned}$$

therefore $$\frac{a + b + c - a' - b' - c'}{abc - a'b'c'} = \frac{1}{b'c'\cos^2\alpha + c'a'\cos^2\beta + a'b'\cos^2\gamma},$$

which is equal to the product of the squares of the axes of the section of

$$a'x^2 + b'y^2 + c'z^2 = 1 \text{ by } x\cos\alpha + y\cos\beta + z\cos\gamma = 0,$$

(SALMON's *Geom. of Three Dim.*, Art. 97).

R and $\cos^2\alpha$, $\cos^2\beta$, $\cos^2\gamma$ are easily found from the above equations which are linear in those quantities, and so the equation to the plane

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0.$$

4721. (Professor SYLVESTER.)—Prove that every point in the plane carried round by the connecting-rod in Watts' or any other kind whatever of three-bar motion has in general three nodes, and that its inverse in respect to each of them is a unicircular quartic.

Solution.

Let AB, BC, CD be the three bars of a three-bar system, where $AB = a$, $BC = b$, $CD = c$, $DA = d$, and P be a point rigidly connected with BC, and h and k, respectively, the perpendiculars from P upon BC and the portion of BC intercepted between B and this perpendicular. Then, if $BAD = \phi$, and θ be the angle which BC makes with AD, taking A as origin of rectangular coordinates, and AD as axis of x, the coordinates of B are $a\cos\phi$ and $a\sin\phi$, and those of C,

$$a\cos\phi + b\cos\theta, \text{ and } a\sin\phi + b\sin\theta;$$

therefore $\quad c^2 = (a\cos\phi + b\cos\theta - d)^2 + (a\sin\phi + b\sin\theta)^2,$

or $2a\cos\phi\,(b\cos\theta - d) + 2a\sin\phi \,.\, b\sin\theta = c^2 + 2bd\cos\theta - a^2 - b^2 - d^2 \ldots (1).$

And, if (x, y) denote the point P,

$$x = a\cos\phi + k\cos\theta - h\sin\theta, \text{ and } y = a\sin\phi + k\sin\theta + h\cos\theta \ldots (2),$$

therefore $\quad 2(kx + hy)\cos\theta + 2(ky - hx)\sin\theta = x^2 + y^2 + h^2 + k^2 - a^2 \ldots (3),$

and from (1) and (2),

$$2(bx + dk - bd)\cos\theta + 2(by - dh)\sin\theta = c^2 + 2bk + 2dx - a^2 - b^2 - d^2 \ldots (4);$$

from (3) and (4), $4\left\{(by-dh)(kx+hy)-(bx+dk-bd)(ky-hx)\right\}^2$

$$=\left\{(by-dh)(x^2+y^2+h^2+k^2-a^2)-(ky-hx)(c^2+2bk+2dx-a^2-b^2-d^2)\right\}^2$$
$$+\left\{(bx+dk-bd)(x^2+y^2+h^2+k^2-a^2)\right.$$
$$\left.-(kx+hy)(c^2+2bk+2dx-a^2-b^2-d^2)\right\}^2 \;\ldots\ldots\; (5),$$

which is the equation to the locus of P. This evidently has double points, where

$$\frac{by-dh}{ky-hx}=\frac{b\;\;+dk-bd}{kx+hy}=\frac{c^2+2bk+2dx-a^2-b^2-d^2}{x^2+y^2+h^2+k^2-a^2}=p, \text{ suppose};$$

therefore $$p=\frac{dhx+dy(b-k)+h(c^2+2bk-a^2-b^2-d^2)}{h(h^2+k^2-a^2)}$$
$$=\frac{by-dh}{ky-hx}=\frac{bx+dk-bd}{kx+hy}\;\ldots\ldots\; (6);$$

and, eliminating x and y from the equations (6), a cubic is obtained to determine p, and each value of p gives one node, and there are therefore three.

The equation (5) to the locus of P is easily reduced to the form

$$4\left\{bh(x^2+y^2)-bdhx-d(h^2+k^2-bk)y\right\}^2$$
$$=\left\{b^2(x^2+y^2)-2bdhy-2bd(b-k)x+d^2(h^2+b^2+k^2-2bk)\right\}$$
$$\times\left\{x^2+y^2+h^2+k^2-a^2\right\}^2-2\left\{bk(x^2+y^2)-bdhy+d(h^2+k^2-bk)y\right\}$$
$$\times\left\{x^2+y^2+h^2+k^2-a^2\right\}\left\{c^2+2bk+2dx-a^2-b^2-d^2\right\}$$
$$+(h^2+k^2)(x^2+y^2)(c^2+2bk+2dx-a^2-b^2-d^2)^2,$$

which meets any circle at its intersections with a cubic, and therefore six times at infinity, *i.e.*, it is a tricircular sextic.

And, if the origin be at one of the nodes (and one must be real, since the cubic in p must have a real root), the equation will be

$$U_0(x^2+y^2)^3+U_1(x^2+y^2)^2+U_2(x^2+y^2)+U_3+V_2=0,$$

where U_0, U_1, U_2, U_3 are homogeneous functions of x and y, of order 0, 1, 2, 3, respectively, and V_2 one of order 2; therefore the equation to the inverse will be $\quad U_0+U_1+U_2+U_3+V_2(x^2+y^2)=0,$

a circular quartic.

4828. (The Editor.)—If the corner of a page of breadth a is turned down in every possible way, so as just to reach the opposite side; (1) show that the mean value of the lengths of the crease is

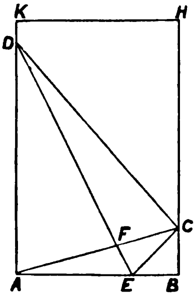
$$\tfrac{1}{8}\left\{7\sqrt{2}+\log(1+\sqrt{2})\right\}a,$$

and (2) the mean area of the part turned down is $\frac{1}{1}\frac{1}{8}a^2$.

Solution.

1. Let ABHK be the page, and ADE the corner turned down into the position DCE, meeting the edge BH in C, and let x, y be the lengths intercepted upon the edges AB, AK from the corner to the crease; then

we have AB $= a$, AE $= x$, AD $= y$, and (C being the point where the corner A rests after folding) if AC cut DE in F, the angles at F are right angles; therefore



$$a^2 = AB^2 = AC^2 - BC^2 = 4AF^2 - CE^2 + BE^2$$

$$= 4\,\frac{x^2y^2}{x^2+y^2} - x^2 + (a-x)^2;$$

therefore $\quad 0 = 4x^2y^2 - 2ax\,(x^2+y^2),$

and $\quad 2xy^2 = a\,(x^2+y^2),$

and the crease $= (x^2+y^2)^{\frac{1}{2}} = \left(x^2 + \dfrac{ax^2}{2x-a}\right)^{\frac{1}{2}}$

$$= x\left(\frac{2x}{2x-a}\right)^{\frac{1}{2}}.$$

Now x may have any value from $x = \frac{1}{2}a$ to $x = a$; hence the required mean value

$$= \frac{1}{\frac{1}{2}a}\int_{\frac{1}{2}a}^{a}\left(\frac{2x^3}{2x-a}\right)\,dx = \frac{1}{a}\int_0^c (c^2+u^2)^{\frac{3}{2}}\,du, \text{ if } c^2 = a, \text{ and } 2x-a = u^2,$$

$$= \frac{2c^4}{16a}\left\{7\sqrt{2} + \log\,(1+\sqrt{2})\right\} = \frac{a}{8}\left\{7\sqrt{2} + \log\,(1+\sqrt{2})\right\}.$$

2. Again, the area turned down $= \frac{1}{2}xy$, hence the required mean value

$$= \frac{1}{\frac{1}{2}a}\int_{\frac{1}{2}a}^{a} x^2\left(\frac{a}{2x-a}\right)^{\frac{1}{2}} dx = \frac{1}{4\sqrt{a}}\int_0^a \frac{(a+z)^2}{\sqrt{z}}\,dz, \text{ if } 2x-a = z, = \frac{14a^2}{15}.$$

6391. (J. J. Walker, M.A.)—If O, A, B, C, D are any five points in space, prove that lines drawn from the middle points of BC, CA, AB respectively parallel to the connectors of D with the middle points of OA, OB, OC, meet in one point E, such that DE passes through, and is bisected by, the centroid of the tetrahedron OABC. [Quest. 6220 is a special case, in *two* dimensions, of the foregoing theorem in *three* dimensions.]

Solution.

This may be generalised into—If ABCD... be a simplicissimum in space of n dimensions, and B_1, C_1, D_1 ... the mid-points of the lines drawn from any point P to B, C, D ... (all the vertices but A), and if parallels to AB_1, AC_1, AD_1 ... be drawn through the centroids of CDE ..., BDE..., BCE..., &c., (the faces of BCDE ...) respectively: these will all meet in the same point Q, AQ will pass through the centroid of PBCD ..., where it will be divided in the ratio of $n-1 : 2$.

Any straight line through (λ', μ', ν' ...) may be represented by

$$\frac{\lambda-\lambda'}{a} = \frac{\mu-\mu'}{b} = \frac{\nu-\nu'}{c} = \ldots,$$

where $a+b+c \ldots = 0$.

And the parallel through (λ'', μ'', ν'' ...) is $\dfrac{\lambda-\lambda''}{a} = \dfrac{\mu-\mu''}{b} = \dfrac{\nu-\nu''}{c} = \ldots.$

If, then, ABCD ... be the simplicissimum of reference, and P $(\lambda', \mu', \nu' \ldots)$, $B_1, C_1, D_1 \ldots$ are $\{\frac{1}{2}\lambda', \frac{1}{2}(\mu'+V), \frac{1}{2}\nu' \ldots\}$ $\{\frac{1}{2}\lambda', \frac{1}{2}\mu', \frac{1}{2}(\nu'+V) \ldots\}$, &c., and the equations to AB_1, AC_1, ... are

$$\frac{\lambda-V}{\lambda'-2V} = \frac{\mu}{\mu'+V} = \frac{\nu}{\nu'} = \frac{\pi}{\pi'} = \ldots, \qquad \frac{\lambda-V}{\lambda'-2V} = \frac{\mu}{\mu'} = \frac{\nu}{\nu'+V} = \frac{\pi}{\pi'} = \ldots,$$

&c. &c. &c.,

and those to the parallels through the centroids of (C, D, E ...) (B, D, E ...), *i.e.*, through

$$\left(0, \; 0, \; \frac{V}{n-1}, \; \frac{V}{n-1} \ldots\right), \quad \left(0, \; \frac{V}{n-1}, \; 0, \; \frac{V}{n-1} \ldots\right), \text{\&c.},$$

are
$$\frac{\lambda}{\lambda'-2V} = \frac{\mu}{\mu'+V} = \frac{\nu - V/(n-1)}{\nu'} = \frac{\pi - V/(n-1)}{\pi'} = \ldots,$$

$$\frac{\lambda}{\lambda'-2V} = \frac{\mu - V/(n-1)}{\mu'} = \frac{\nu}{\nu'+V} = \frac{\pi - V/(n-1)}{\pi'} = \ldots,$$

&c. &c. &c.,

which all meet where

$$\frac{\lambda}{\lambda'-2V} = \frac{\mu - V/(n-1)}{\mu'} = \frac{\nu - V/(n-1)}{\nu'} = \ldots = \left(\frac{1}{n-1}\right),$$

so that, if $(\lambda'', \mu'', \nu'' \ldots)$ be the point of intersection Q,

$$\lambda'' = \frac{\lambda'}{n-1} - 2\,\frac{V}{n-1}, \quad \mu'' = \frac{\mu'}{n-1} + \frac{V}{n-1}, \quad \nu'' = \frac{\nu'}{n-1} + \frac{V}{n-1}, \text{\&c.},$$

and the equations to AQ are $\dfrac{\lambda - V}{\lambda''-V} = \dfrac{\mu}{\mu''} = \dfrac{\nu}{\nu''} = \ldots,$

or
$$\frac{\lambda - V}{\lambda' - (n+1)V} = \frac{\mu}{\mu'+V} = \frac{\nu}{\nu'+V} = \ldots,$$

and the line passes through $\left(\dfrac{\lambda'}{n+1}, \dfrac{\mu'+V}{n+1}, \dfrac{\nu'+V}{n+1} \ldots\right)$ or $(\bar{\lambda}, \bar{\mu}, \bar{\nu} \ldots)$, the centroid of (PBCD ...); and, since

$$(n+1)\,\bar{\lambda} = (n-1)\,\lambda'' + 2V, \quad (n+1)\,\bar{\mu} = (n-1)\,\mu'' + 2 \times 0,$$

$$(n+1)\,\bar{\nu} = (n-1)\,\nu'' + 2 \times 0, \text{ \&c.},$$

this centroid divides AB in the ratio $n-1 : 2$.

[Solutions by Professors Minchin and Genese are given in Vol. xxxiv., p. 40.]

7131. (W. J. C. Sharp, M.A.)—Prove that the vector equations to the centrodes of a three-bar motion, which are easily derived from one another by a linear substitution, are of the third degree in the vectors, and reduce to the second where the algebraical perimeter of the figure is zero.

Solution.

If AB, BC, CD be the bars, a, b, c their lengths, and $AD = d$ the distance between the fixed ends, the instantaneous centre is O, the intersection of the lines AB and CD. (CLIFFORD, *Dynamic*, p. 166.)

It then follows at once that, if $AO = r$ and $BO = r'$,

$$2\cos AOD = \frac{r^2+r'^2-d^2}{rr'} = \frac{(r-a)^2+(r'-c)^2-b^2}{(r-a)(r'-c)},$$

or

$$\frac{(r+r')^2-d^2}{rr'} = \frac{(r+r'-a-c)^2-b^2}{(r-a)(r'-c)},$$

which is the vector equation to the fixed centrode, and is, in general, of the third order. If, however, $r+r' \pm d = (r+r'-a-c) \pm b$, *i.e.*, $a+c = \pm(b \pm d)$, a factor will divide out, and the equation is of the second order.

If $BO = R$, and $CO = R'$ the relation between R and R′ will determine the movable centrode, and it is easily seen that this is derived from the above by putting $r = R+a$, and $r' = R'+c$;

and the equation to this centrode is of the same order as that to the other, viz.,

$$\frac{(R+R'-a-c)^2-d^2}{(R-a)(R'-c)} = \frac{(R+R')^2-b^2}{R \cdot R'},$$

which may therefore be deduced from that to the fixed centrode by writing b for d and *vice versâ*.

8177. (B. HANUMANTA RAU, B.A.)—The images of the circumcentre of a triangle ABC with respect to the sides are A′, B′, C′; prove (1) that the triangles A′B′C′ and ABC are equal; (2) that they have the same nine-point circle. Find the equation of the circumcircle of A′B′C′ and the angle at which the two circumcircles cut each other.

Solution.

If a, b, c be the middle points of the sides, evidently B′C′ is parallel to and double of bc (Euclid, VI., 2), and therefore parallel and equal to BC, and so for the other sides of A′B′C′, and this triangle is equal to ABC in all respects. Also O, the circumcentre of ABC, is the orthocentre of A′B′C′, since A′O, B′O, and C′O are perpendicular to BC, CA, and AB, and therefore to B′C′, C′A′, and A′B′, respectively; and a, b, c are the middle points of the portions of the perpendiculars intercepted between the orthocentre and the vertices, therefore the circle through abc, the nine-point circle of ABC, is also the nine-point circle of A′B′C′. The relation of the triangles ABC and A′B′C′ is mutual, for, since $BC' = BO = BA'$, the perpendicular from B on AC or A′C′ bisects A′C′, and the orthocentre of ABC is the circumcentre of A′B′C′.

If O′ be this point, the angle at which circumcircles cut is

$$2\sin^{-1}\frac{OO'}{2R} = \cos^{-1}\left(1-\frac{OO'^2}{2R^2}\right),$$

where R is the circumradius $= \cos^{-1}\tfrac{1}{2}(1-8\cos A\cos B\cos C)$.

If (x', y', z') be the trilinear coordinates of O, the points A′, B′, C′ are $-x', p_2-y', p_3-z'$; $p_1-x', -y', p_3-z'$; $p_1-x', p_2-y', -z'$ respectively, where p_1, p_2, p_3 are the perpendiculars of the triangle ABC, and by substituting these in

$$yz \sin A + zx \sin B + xy \sin C - (x \sin A + y \sin B + z \sin C)(hx + ky + lz) = 0,$$

h, k, and l are easily determined, and the equation to the circumcircle of A′B′C′. The line $hx + ky + lz = 0$, the radical axis of the circumcircles is the line bisecting OO′ at right angles.

The equality of the triangles ABC and A′B′C′ may be otherwise demonstrated in a way which explains the geometrical meaning of the equation to a conic circumscribed to a triangle, and demonstrates an interesting property of such figures.

If P (x', y', z') be any point, A″, B″, C″ its projections on the sides of the triangle of reference, and A′, B′, C′ points on PA″, PB″, PC″, such that

$$\text{PA}' = f.\text{PA}'' = fx', \quad \text{PB}' = g.\text{PB}'' = gy', \quad \text{and PC}' = h.\text{PC}'' = hz';$$

the area of the triangle A′B′C′

$$= \tfrac{1}{2}\{\text{PB}'.\text{PC}' \sin A + \text{PC}'.\text{PA}' \sin B + \text{PA}'.\text{PB}' \sin C\}$$

$$= \tfrac{1}{2}\{gh \sin A\, y'z' + hf \sin B\, z'x' + fg \sin C\, x'y'\} \quad \ldots\ldots\ldots\ldots \quad (1).$$

Now, if $f = g = h = 2$, and x', y', z' be the circumcentre, the points A′, B′, C′ are those in the Question, and

$$\triangle \text{A}'\text{B}'\text{C}' = 2(y'z' \sin A + z'x' \sin B + x'y' \sin C)$$

$$= 4(\triangle bOc + \triangle cOa + \triangle aOb) = \triangle \text{ABC}.$$

Now, returning to equation (1). If P lie upon the circumscribed conic

$$gh \sin A\, yz + hf \sin B\, zx + fg \sin C\, xy = 0,$$

the triangle A′B′C′ vanishes, *i.e.*, A′, B′, C′ lie on a straight line. Or, in other words, if the perpendiculars upon the sides of the triangle of reference from any point on the circumscribed conic

$$\lambda yz + \mu zx + \nu xy = 0$$

be produced to points A′, B′, C′, such that

$$\text{PA}' = f.\text{PA}'', \quad \text{PB}' = g.\text{PB}'', \quad \text{PC}' = h.\text{PC}'',$$

the points A′, B′, and C′ will lie in a line, if

$$f : g : h :: \sin A/\lambda : \sin B/\mu : \sin C/\nu.$$

The well-known property of the Simpson lines is a particular case of this.

8592. (Professor Mathews, M.A.)—Through a point P are drawn three planes, each parallel to a pair of opposite edges of a tetrahedron ABCD. Prove that the 12 finite intersections of these planes with the edges of the tetrahedron lie on the same quadric surface; and that, if $\text{BC}^2 + \text{AD}^2 = \text{CA}^2 + \text{BD}^2 = \text{AB}^2 + \text{CD}^2$ (*i.e.*, if each edge of the tetrahedron is perpendicular to the opposite edge), there is one position of P for which the quadric surface is a sphere.

Solution.

If $\lambda_1, \mu_1, \nu_1, \pi_1$ be the tetrahedral coordinates of P referred to the tetrahedron as tetrahedron of reference, the equations to the three planes

through P parallel to opposite edges of the tetrahedron are

$$\frac{\lambda+\mu}{\lambda_1+\mu_1}=\frac{\nu+\pi}{\nu_1+\pi_1},\quad \frac{\lambda+\nu}{\lambda_1+\nu_1}=\frac{\mu+\pi}{\mu_1+\pi_1},\text{ and }\frac{\lambda+\pi}{\lambda_1+\pi_1}=\frac{\mu+\nu}{\mu_1+\nu_1}.$$

Now, if $\quad A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu+2A_{13}\lambda\nu+\ldots=0$

be any quadric, this meets $\lambda=\mu=0$ at points at which

$$A_{33}\nu^2+2A_{34}\nu\pi+A_{44}\pi^2=0 \qquad\qquad (1);$$

but, by Question, these points lie on

$$\frac{\lambda+\nu}{\lambda_1+\nu_1}=\frac{\mu+\pi}{\mu_1+\pi_1}\text{ and }\frac{\lambda+\pi}{\lambda_1+\pi_1}=\frac{\mu+\nu}{\mu_1+\nu_1},$$

and therefore the equation (1) is equivalent to

$$\left(\frac{\nu}{\lambda_1+\nu_1}-\frac{\pi}{\mu_1+\pi_1}\right)\left(\frac{\nu}{\mu_1+\nu_1}-\frac{\pi}{\lambda_1+\pi_1}\right)=0;$$

therefore $\qquad A_{33}:-2A_{34}:A_{44}$

$$::\frac{1}{(\lambda_1+\nu_1)(\mu_1+\nu_1)}:\frac{1}{(\mu_1+\pi_1)(\mu_1+\nu_1)}+\frac{1}{(\lambda_1+\nu_1)(\lambda_1+\pi_1)}:\frac{1}{(\lambda_1+\pi_1)(\mu_1+\pi_1)},$$

and so on. And the quadric

$$\frac{\lambda^2}{(\lambda_1+\mu_1)(\lambda_1+\nu_1)(\lambda_1+\pi_1)}+\frac{\mu^2}{(\mu_1+\nu_1)(\mu_1+\pi_1)(\mu_1+\lambda_1)}+\ldots$$
$$-\frac{\lambda\mu}{(\lambda_1+\mu_1)}\left\{\frac{1}{(\lambda_1+\nu_1)(\mu_1+\pi_1)}+\frac{1}{(\mu_1+\nu_1)(\lambda_1+\pi_1)}\right\}\ldots=0$$

meets the edges at their finite intersections with the three planes.

If $A_{11}\lambda^2+A_{22}\mu^2+\ldots+2A_{12}\lambda\mu\ldots=0$ be a sphere,

$$\frac{A_{11}+A_{22}-2A_{12}}{(1.2)^2}=\frac{A_{11}+A_{33}-2A_{13}}{(1.3)^2}=\frac{A_{22}+A_{33}-2A_{23}}{(2.3)^2}=\&c.,$$

where (1 . 2), &c. are the edges of the tetrahedron of reference (*Proceedings of Lond. Math. Soc.*, Vol. XVIII., p. 341).

And $\qquad (\nu_1+\pi_1)(\lambda_1+\mu_1+2\nu_1)(\lambda_1+\mu_1+2\pi_1)=r(1.2)^2$, &c.,

or $\quad (\nu_1+\pi_1)\{C^2-(\pi_1-\nu_1)^2\}=r(1.2)^2$, &c., where $\lambda+\mu+\nu+\pi=C$,

and therefore, if $\lambda_1=\mu_1=\nu_1=\pi_1$, *i.e.*, if P be the centroid,

$$(\nu_1+\pi_1)C^2=r(1.2)^2,\quad (\mu_1+\pi_1)C^2=r(1.3)^2,\quad (\mu_1+\nu_1)C^2=r(1.4)^2,$$
$$(\lambda_1+\mu_1)C^2=r(3.4)^2,\quad (\lambda_1+\nu_1)C^2=r(2.4)^2,\quad (\lambda_1+\pi_1)C^2=r(2.3)^2,$$

and $\qquad (1.2)^2+(3.4)^2=(1.3)^2+(2.4)^2=(1.4)^2+(2.3)^2;$

and the tetrahedron is rectangular (see Solution of Question 3228, *Appendix* III., Vol. XLVIII., p. 168).

8940. (W. J. C. SHARP, M.A.)—If

$$S\equiv ax^2+by^2+cz^2+dw^2+2lyz+2mzx+2nxy+2pxw+2qyw+2rzw,$$

and $\qquad P_{1.2}\equiv ax_1x_2+by_1y_2+cz_1z_2+dw_1w_2+l(y_1z_2+y_2z_1)+\&c.;$

show that $\quad S_1S_2S_3+2P_{1.2}P_{2.3}P_{3.1}-S_1P_{2.3}^2-S_2P_{3.1}^2-S_3P_{1.2}^2$

$$\equiv A\begin{vmatrix} y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \end{vmatrix}^2+\ldots+2L\begin{vmatrix} z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \end{vmatrix}\begin{vmatrix} w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix}+\&c.,$$

where A, *u* are the first minors of the discriminant of S.

Solution.

If x, y, z, w be eliminated from S by means of the relations

$$(\lambda+\mu+\nu)\,x = \lambda x_1+\mu x_2+\nu x_3,$$
$$(\lambda+\mu+\nu)\,y = \lambda y_1+\mu y_2+\nu y_3,$$
$$(\lambda+\mu+\nu)\,z = \lambda z_1+\mu z_2+\nu z_3,$$
$$(\lambda+\mu+\nu)\,w = \lambda w_1+\mu w_2+\nu w_3,$$

the result is

$$\frac{1}{(\lambda+\mu+\nu)^2}\left\{\lambda^2S_1+\mu^2S_2+\nu^2S_3+2\lambda\mu P_{12}+2\mu\nu P_{23}+2\nu\lambda P_{31}\right\}.$$

Now, as I have shown in a paper read before the London Mathematical Society in December, 1883, and in a note, *Reprint*, Vol. XLIII., p. 47, λ, μ, ν are areal coordinates of any point (x, y, z, w) on the plane through three points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3) referred to the triangle of which those points are the vertices. And the plane equation to the section of the quadric by the plane is

$$\lambda^2S_1+\mu^2S_2+\nu^2S_3+2\lambda\mu P_{12}+2\mu\nu P_{23}+2\nu\lambda P_{31}=0 \quad\ldots\ldots\ldots\ldots(1).$$

If the plane satisfy the tangential equation to the surface, the section must have a double point. Now the equation to the plane is

$$\begin{vmatrix} x, & y, & z, & w \\ x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{vmatrix}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and therefore $S_1S_2S_3+2P_{23}P_{31}P_{12}-S.P_{23}^2-S_2P_{31}^2-S_3P_{12}^2=0,$

the condition that (1) should have a double point, and

$$A\begin{vmatrix} y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \end{vmatrix}^2+\ldots+2L\begin{vmatrix} z_1, & z_2, & z_3 \\ w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \end{vmatrix}\begin{vmatrix} w_1, & w_2, & w_3 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix}+\ldots=0,$$

the condition that (2) should touch the surface, hold simultaneously. The sinisters are both of the same order in the coefficients and variables, so they can only differ by a factor which is easily found to be unity.

[Mr. EDWARDES sends the following solution:—As in the solution by 8970, we have

$$\left\|\begin{matrix} \dfrac{dS_1}{dx_1}, & \ldots & \ldots & \dfrac{dS_1}{dw_1} \\ \ldots, & \ldots & \ldots & \ldots \\ \dfrac{dS_3}{dx_3}, & \ldots & \ldots & \dfrac{dS_3}{dw_3} \end{matrix}\right\|\left\|\begin{matrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{matrix}\right\| = 16\begin{vmatrix} S_1, & P_{12}, & P_{13} \\ P_{12}, & S_2, & P_{23} \\ P_{13}, & P_{23}, & S_3 \end{vmatrix}$$

and writing $S \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw,$

the same is
$$\Sigma \begin{vmatrix} \frac{dS_1}{dx_1}, & \frac{dS_1}{dy_1}, & \frac{dS_1}{dz_1} \\ \frac{dS_2}{dx_2}, & \frac{dS_2}{dy_2}, & \frac{dS_2}{dz_2} \\ \frac{dS_3}{dx_3}, & \frac{dS_3}{dy_3}, & \frac{dS_3}{dz_3} \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix},$$

or
$$\Sigma \left\|\begin{matrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \end{matrix}\right\| \left\|\begin{matrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{matrix}\right\| \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$

$$= \Sigma \left\{ D \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} + L \begin{vmatrix} y_1, & z_1, & w_1 \\ y_2, & z_2, & w_2 \\ y_3, & z_3, & w_3 \end{vmatrix} + M \begin{vmatrix} z_1, & w_1, & x_1 \\ z_2, & w_2, & x_2 \\ z_3, & w_3, & x_3 \end{vmatrix} + N \begin{vmatrix} w_1, & x_1, & y_1 \\ w_2, & x_2, & y_2 \\ w_3, & x_3, & y_3 \end{vmatrix} \right\} \times \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$

= result.]

8969. (W. J. C. SHARP, M.A.)—If the ternary n-ic be written
$$ax^n + \frac{n}{1}(b_1y + b_2z)x^{n-1} + \frac{n(n-1)}{1.2}(c_1y^2 + 2c_2y^2 + c_3z^2)x^{n-2} + \text{\&c.},$$
and $ax + b_1y + b_2z$ be written for a,
$b_1x + c_1y + c_2z$ be written for b_1,
$b_2x + c_2y + c_3z$ be written for b_2, and so on,
in any invariant or covariant; the result will be a covariant of the $(n+1)$-ic $\quad ax^{n+1} + \frac{n+1}{1}(b_1y + b_2z)x^n + \text{\&c.}$

Solution.

For any covariant of the $(n+1)$-ic, the operation $y\frac{d}{dx}$ must be equivalent to
$$a\frac{d}{db_1} + 2b_1\frac{d}{dc_1} + b_2\frac{d}{dc_2} + 3c_1\frac{d}{dd_1} + 2c_2\frac{d}{dd_2} + c_3\frac{d}{dd_3} + \text{\&c.}$$
Writing the $(n+1)$-ic,
$$u \equiv a'x^n + \frac{n}{1}(b'_1y + b'_2z)x^{n-1} + \text{\&c., where } a' = ax + b_1y + b_2z, \text{ \&c.,}$$

as in question, we have

$$y\frac{du}{dx} = y\left\{(n+1)\,ax^n + n+1\,.\,n\,(b_1y+b_2z)\,x^{n-1}\right.$$
$$\left.+\frac{n+1\,.\,n\,.\,n-1}{2}\,(c_1y^2+2c_2yz+c_3z^2)\,x^{n-2}+\&c.\right\}.$$

Then $$\left(a\frac{d}{db_1}+2b_1\frac{d}{dc_1}+b_2\frac{d}{dc_2}+\&c.\right)u$$

$$= a\,(x^ny+nx^ny)+2b_1\left(nx^{n-1}y^2+\frac{n\,.\,n-1}{2}\,x^{n-1}y^2\right)$$
$$+\,b_2\,(nx^{n-1}yz+nzx^{n-1}y+n\,.\,n-1\;yz\,x^{n-1})$$
$$+\,3c_1\left(\frac{n\,.\,n-1}{2}\,x^{n-2}y^3+\frac{n\,.\,n-1\,.\,n-2}{3\,!}\,x^{n-2}y^3\right)$$
$$+\,2c_2\left(n\,.\,n-1\;y^2zx^{n-2}+\frac{n\,.\,n-1}{2}\,x^{n-2}y^2z+\frac{n\,.\,n-1\,.\,n-2}{2}\,x^{n-2}y^2z\right)$$
$$+\,c_3\left(\frac{n\,.\,n-1}{2}\,x^{n-2}\,(z^2y+2yz^2)+\frac{n\,.\,n-1\,.\,n-2}{2}\,yz^2x^{n-2}\right)+\&c.$$
$$= (n+1)\,ax^ny+n+1\,.\,n\,(b_1y+b_2z)\,x^{n-1}y$$
$$+\,\frac{n+1\,.\,n\,.\,n-1}{2}\,(c_1y^2+2c_2yz+c_3z^2)\,x^{n-2}y+\&c.$$
$$= y\left\{(n+1)\,ax^n+n+1\,.\,n\,(b_1y+b_2z)\,x^{n-1}\right.$$
$$\left.+\,\frac{n+1\,.\,n\,.\,n-1}{2}\,(c_1y^2+2c_2yz+c_3z^2)\,x^{n-2}+\&c.\right\} = y\frac{du}{dx} \text{ from above.}$$

Similarly, $z\dfrac{d}{dx}$ is equivalent to

$$a\frac{d}{db_2}+b_1\frac{d}{dc_2}+2b_2\frac{d}{dc_3}+c_1\frac{d}{dd_2}+2c_2\frac{d}{dd_3}+3c_3\frac{d}{dd_4}+\&c.,$$

and therefore, &c.

[Any invariant or covariant of a quantic may be expressed as a function of the differential coefficients of the quantic, and this same function of its differential coefficients will be a concomitant of any other quantic. Now, if u_n and u_{n+1} denote the given quantic and the one derived from it by making the proposed substitutions, $d^{p+q+r}\,u_{n+1}/\,dx^p\,.\,dy^q\,.\,dz^r$ is the result of making the same substitutions in $d^{p+q+r}\,u_n/\,dx^p\,.\,dy^q\,.\,dz^r$, and the symbolical form of the transformed concomitant is the same as that of the original one; therefore, &c. This may, of course, be extended to k-ary quantics, and to substitutions of a higher order in the variables.]

8970. (W. J. C. SHARP, M.A.)—If X, Y ... U denote the determinants

$$\left\|\begin{matrix} x_1, & y_1, & z_1, & w_1, & u_1 \\ x_2, & y_2, & z_2, & w_2, & u_2 \\ x_3, & y_3, & z_3, & w_3, & u_3 \\ x_4, & y_4, & z_4, & w_4, & u_4 \end{matrix}\right\|,$$

and V_1, V_2, V_3, V_4 be the values of the quinary quadratic V when $(x_1, y_1, z_1, w_1, u_1)$, $(x_2, y_2, z_2, w_2, u_2)$, &c. are put for (x, y, z, w, u), and $S_{1.2}$, &c. stand for $\frac{1}{2}\left(x_1\frac{d}{dx_2}+y_1\frac{d}{dy_2}+\ldots\right)V_2$, &c.,

$$\begin{vmatrix} V_1, & S_{1.2}, & S_{1.3}, & S_{1.4} \\ S_{1.2}, & V_2, & S_{2.3}, & S_{2.4} \\ S_{1.3}, & S_{2.3}, & V_3, & S_{3.4} \\ S_{1.4}, & S_{2.4}, & S_{3.4}, & V_4 \end{vmatrix} = AX^2+BY^2+\&c.,$$

where A, B, &c. are the first minors of the discriminant of V.

Solution.

This question is a generalisation of 8960, which is itself one of the property (otherwise) proved in SALMON's *Conics*, Ed. 5, Art. 294, and which is the analytical ground of the theory of reciprocal polars.

The proof in this case is the same as that in 8940; for, if x, y, z, w, u be eliminated from $V = 0$, by means of the equations

$$(\lambda+\mu+\nu+\pi)\,x = \lambda x_1+\mu x_2+\nu x_3+\pi x_4,$$
$$(\lambda+\mu+\nu+\pi)\,y = \lambda y_1+\mu y_2+\nu y_3+\pi y_4,$$
$$\&c. \qquad \&c. \qquad \&c.$$

the result will be the equation to the section of $V = 0$ by the linear locus through $(x_1, y_1, z_1, w_1, u_1)$, $(x_2, y_2, z_2, w_2, u_2)$, $(x_3\ldots)$, $(x_4\ldots)$, and, if this locus have a double point, the determinant in the question, the discriminant of the locus of section, will vanish, and also the result of substituting X, Y, ... U, in the reciprocal equation; and those two quantities being each of the same order in the coefficients of V, and in x, y, ... &c., can only differ by a factor which will be found to be unity. The property may easily be extended to space of any dimensions (or k-ary quadratics) and proves that the theory of reciprocal polars holds for space of all dimensions.

[Mr. EDWARDES sends the following solution:—

Since $\quad x_1\frac{dV_1}{dx_1}+y_1\frac{dV_1}{dy_1}+\&c. = 2V_1$, we have

$$\left\| \begin{matrix} \frac{dV_1}{dx_1}, & \frac{dV_1}{dy_1}, & \frac{dV_1}{dz_1}, & \frac{dV_1}{dw_1}, & \frac{dV_1}{du_1} \\ \frac{dV_2}{dx_2}, & \frac{dV_2}{dy_2}, & \frac{dV_2}{dz_2}, & \frac{dV_2}{dw_2}, & \frac{dV_2}{du_2} \\ \frac{dV_3}{dx_3}, & \frac{dV_3}{dy_3}, & \frac{dV_3}{dz_3}, & \frac{dV_3}{dw_3}, & \frac{dV_3}{du_3} \\ \frac{dV_4}{dx_4}, & \frac{dV_4}{dy_4}, & \frac{dV_4}{dz_4}, & \frac{dV_4}{dw_4}, & \frac{dV_4}{du_4} \end{matrix} \right\| \quad \left\| \begin{matrix} x_1, & y_1, & z_1, & w_1, & u_1 \\ x_2, & y_2, & z_2, & w_2, & u_2 \\ x_3, & y_3, & z_3, & w_3, & u_3 \\ x_4, & y_4, & z_4, & w_4, & u_4 \end{matrix} \right\|$$

$$= \begin{vmatrix} 2V_1, & 2S_{21}, & 2S_{31}, & 2S_{41} \\ 2S_{12}, & 2V_2, & 2S_{32}, & 2S_{42} \\ 2S_{13}, & 2S_{23}, & 2V_3, & 2S_{43} \\ 2S_{14}, & 2S_{24}, & 2S_{34}, & 2V_4 \end{vmatrix}$$

and obviously $S_{12} = S_{21}$, &c., therefore this determinant becomes

$$16\begin{vmatrix} V_1, & S_{12}, & S_{13}, & S_{14} \\ S_{12}, & V_2, & S_{23}, & S_{24} \\ S_{13}, & S_{23}, & V_3, & S_{34} \\ S_{14}, & S_{24}, & S_{34}, & V_4 \end{vmatrix}.$$

But expanding in another way, we have

$$\Sigma\begin{vmatrix} \frac{dV_1}{dx_1}, & \frac{dV_1}{dy_1}, & \frac{dV_1}{dz_1}, & \frac{dV_1}{dw_1} \\ \frac{dV_2}{dx_2}, & \frac{dV_2}{dy_2}, & \&c. & \\ \frac{dV_3}{dx_3}, & \&c. & & \\ \frac{dV_4}{dx_4}, & \&c. & & \end{vmatrix} \cdot \begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix},$$

and this $= 16\Sigma \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_3 \end{vmatrix}^2,$

viz., $16\Sigma AX^2$, the quadratic being written

$$(a\,b\,c\,d\,e\,f\,g\,h\,l\,m\,n\,p\,q\,r\,s \between x\,y\,z\,w\,u)^2.]$$

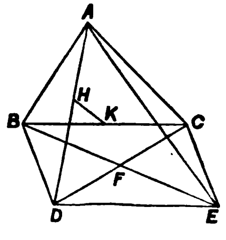
9561. (W. J. C. Sharp, M.A.)—If (1 . 2), (2 . 3), &c. denote the edges of a tetrahedron, and D_1, D_2, D_3 the shortest distances, and θ_1, θ_2, θ_3 the angles between (2 . 3) and (1 . 4), (3 . 1) and (2 . 4), and (1 . 2) and (3 . 4), respectively; prove that

(1) $\cos\theta_1 = \dfrac{1}{2\,(2\,.\,3)(1\,.\,4)}\left\{(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right\}$, &c., &c.,

and (2) the square of the volume

$$=\frac{D_1^2}{144}\left\{4\,(2\,.\,3)^2\,(1\,.\,4)^2-\left[(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right]^2\right\}=\&c., \&c.$$

Solution.



Let ABCD be a tetrahedron. From D draw DE parallel and equal to BC; join BE and AE.

BCDE is a parallelogram and its diagonals bisect each other (in F).

Now $\theta_1 = ADE$;

therefore $\cos\theta_1 = \cos ADE$

$$=\frac{1}{2AD\,.\,DE}\left\{AD^2+DE^2-AE^2\right\}$$

$$=\frac{1}{2AD\,.\,BC}\left\{AD^2+BC^2-AE^2\right\};$$

but $$AB^2+AE^2 = 2BF^2+2AF^2,$$
$$DA^2+AC^2 = 2DF^2+2AF^2, \quad DB^2+BC^2 = 2BF^2+2DF^2,$$

therefore $$DA^2+DB^2+AC^2+BC^2 = 2BF^2+2AF^2+4DF^2$$
$$= AB^2+AE^2+CD^2,$$

therefore $$AD^2+BC^2-AE^2 = AB^2+CD^2-DB^2-AC^2,$$

and therefore $$\cos\theta_1 = \frac{1}{2AD\,.\,BC}\left\{AB^2+CD^2-DB^2-AC^2\right\}$$
$$= \frac{1}{2\,(1\,.\,4)\,(2\,.\,3)}\left\{(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right\},$$

and so for $\cos\theta_2$ and $\cos\theta_3$.

From these values it follows that the opposite edges are at right angles, if $\quad (1\,.\,2)^2+(3\,.\,4)^2 = (1\,.\,3)^2+(2\,.\,4)^2 = (1\,.\,4)^2+(2\,.\,3)^2$, and conversely. If HK be the shortest distance D_1, it is at right angles to AD and BC, and therefore to AD and DE, and so to the plane ADE; and since BC is parallel to that plane, it is equal to the perpendicular from C upon the plane ADE.

Now $V \equiv$ tetrahedron ABCD = tetrahedron ACDE

$$= \frac{AD\,.\,BC}{6}\sin\theta_1 \times \text{perpendicular from C to ADE}$$
$$= \frac{AD\,.\,BC}{6}\sin\theta_1\,.\,D_1;$$

therefore $$V^2 = \frac{AD^2\,.\,BC^2}{36}D_1^2\,(1-\cos^2\theta_1)$$
$$= \frac{1}{144}\left\{4\,(1\,.\,4)^2\,(2\,.\,3)^2-\left[(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right]^2\right\} = \text{\&c.}$$

From the above $$(2\,.\,3)\,(1\,.\,4)\,D_1\sin\theta_1 = (3\,.\,1)\,(2\,.\,4)\,D_2\sin\theta_2$$
$$= (1\,.\,2)\,(3\,.\,4)\,D_3\sin\theta_3.$$

[Mr. Edwardes sends the following solution:—Denote the sides of the tetrahedron $(1\,.\,4)$, $(3\,.\,4)$, $(2\,.\,4)$, $(2\,.\,3)$, $(1\,.\,2)$, $(1\,.\,3)$, by a, b, c, d, e, f, respectively. Through the point (c, b, d), on the plane of abc, draw a parallel to a, and let angle $(a, d) = \theta_1$, &c., also, let angles (e, c), (e, a), (c, a), (c, d), (cde, abc) be $\lambda, \mu, \nu, \alpha, \phi$, respectively. Then we have

$$-\cos\theta_1 = \cos\alpha\cos\nu-\sin\alpha\sin\nu\cos\phi, \quad\text{and}\quad \cos\phi = \frac{\cos\mu-\cos\lambda\cos\nu}{\sin\lambda\sin\nu},$$

therefore $$-\cos\theta_1 = \cos\nu\left(\cos\alpha+\frac{\sin\alpha\cos\lambda}{\sin\lambda}\right)-\frac{\sin\alpha}{\sin\lambda}\cos\mu$$
$$= \frac{a^2+c^2-b^2}{2ac}\left(\frac{d^2+c^2-e^2}{2dc}+\frac{e}{d}\,\frac{e^2+c^2-d^2}{2ec}\right)-\frac{e}{d}\,\frac{e^2+a^2-f^2}{2ae}$$
$$= \frac{a^2+c^2-b^2}{2ad}-\frac{e^2+a^2-f^2}{2ad} = \frac{c^2+f^2-b^2-e^2}{2ad},$$

hence we have $$\cos\theta_1 = \frac{b^2+e^2-c^2-f^2}{2ad}$$
$$= \frac{1}{2\,(2\,.\,3)(1\,.\,4)}\left\{(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right\}, \text{ \&c., \&c.}$$

Again, $D_1 = \dfrac{6V}{ad\sin\theta_1}$; therefore, $V = \dfrac{D_1}{6}\,ad\sin\theta_1$;

therefore, $V^2 = \dfrac{D_1^2}{36}a^2d^2\sin^2\theta_1 = \dfrac{D_1^2}{36}a^2d^2\left\{1-\left(\dfrac{b^2+e^2-c^2-f^2}{2ad}\right)^2\right\}$

$= \dfrac{D_1^2}{144}\left\{4a^2d^2-(b^2+e^2-c^2-f^2)^2\right\}$

$= \dfrac{D_1^2}{144}\left\{4\,(2\,.\,3)^2(1\,.\,4)^2-\left[(1\,.\,2)^2+(3\,.\,4)^2-(2\,.\,4)^2-(1\,.\,3)^2\right]^2\right\}$ = &c., &c.]

7384. (Professor S. Réalis.)—Étant donnée la série illimitée 7, 13, 25, 43, 67, 97, 133, 137, ..., dont le terme général, celui qui en a n avant lui, est $A_n = 3(n^2+n)+7$: démontrer les propositions suivantes :—(1) sur cinq termes consécutifs, pris à volonté dans la série, un terme est divisible par 5 ; (2) sur sept termes consécutifs, deux sont divisibles par 7 ; (3) sur treize termes consécutifs, deux sont divisibles par 13 ; (4) aucun terme de la série n'est égal à un cube; (5) une infinité de termes, tels que $A_3 = 25$, $A_{37} = 4225$, etc., sont des carrés divisibles par 25 ; (6) la deuxième et la troisième proposition sont comprises, comme cas particuliers, dans la suivante : si N est un nombre premier, de la forme $6m+1$, sur N termes consécutifs de la série, deux sont divisibles par N ; (7) on peut affirmer aussi que, à l'exception de 5. aucun nombre premier de la forme $6m-1$ ne peut diviser aucun terme de la série.

Solution.

Out of any five consecutive numbers one must be of the form $5p+2$, and the corresponding term of the series will be

$$3(5p+2)(5p+3)+7 = 25(3p^2+3p+1),$$

which proves (1)—indeed, that one term in five is divisible by 5^2—and (5), since an infinite series of squares of the form $3p^2+3p+1$ can be found [see Solution of Question 4535, vol. xxiii., pp. 30, 99]. Similarly, out of seven consecutive values of n two are of the forms $7p$ and $7p+6$, and in each case $3(n^2+n)+7$ is divisible by 7 (2) ; and out of thirteen, two are of the forms $13p+1$ and $13p+11$, which each of them makes $3(n^2+n)+7$ divisible by 13 (3). No term is an exact cube, for if

$$3(n^2+n)+7 \equiv 3(n^2+n+2)+1$$

be a cube at all, it is the cube of a number of the form $3p+1$, and n^2+n+2 must be divisible by 3, which it never is (4).

Again, $4A_n \equiv 3(4n^2+4n+1)+25 \equiv 3p^2+25$

if $p = 2n+1$, and if $6m+1$ be a prime, $3p^2+25 = (6m+1)y$ has real solutions. If $6m-1$ be a prime, $3p^2+25 = (6m-1)y$ has not, except in the special case $m = 1$ (7) ; and if, when $n = q$, $3(n^2+n)+7$ is divisible by $6m+1$, it is so when $n = 6m-q$ or $(6m+1)p+q$ or $(6m+1)p+6m-q$, which proves (6).

It may be interesting to point out that

$$\Sigma_0^\infty A_n x^n \equiv (7-8x-7x^2)(1-x)^{-3}.$$

APPENDIX II.

NEW QUESTIONS. By W. J. Curran Sharp, M.A.

9776. If perpendiculars p, q, r be drawn from the vertices of a triangle upon any tangent to the circumcircle, these are connected by the relation

$$\begin{vmatrix} o, & p, & q, & r \\ p, & o, & c^2, & b^2 \\ q, & c^2, & o, & a^2 \\ r, & b^2, & a^2, & o \end{vmatrix} = 0.$$

Prove this, and show that a similar relation holds in space of n dimensions.

9777. If $l\lambda + m\mu + n\nu + \ldots = 0$ be the equation to a linear locus in space of n dimensions, in terms of the simplicissimum content coordinates (areal, tetrahedral, &c.), (see Question 8242); show that l, m, n, &c. are proportional to the perpendiculars drawn from the vertices of the simplicissimum of reference upon the locus.

9778. If the variables α, β, γ, δ be removed from the tangential equation to a surface, by substitution from

$$(\lambda+\mu+\nu)\,\alpha = \lambda\alpha_1 + \mu\alpha_2 + \nu\alpha_3,\quad (\lambda+\mu+\nu)\,\beta = \lambda\beta_1 + \mu\beta_2 + \nu\beta_3,$$
$$(\lambda+\mu+\nu)\,\gamma = \lambda\gamma_1 + \mu\gamma_2 + \nu\gamma_3,\quad (\lambda+\mu+\nu)\,\delta = \lambda\delta_1 + \mu\delta_2 + \nu\delta_3;$$

show that the resulting equation in (λ, μ, ν) is a tangential equation to the tangent cone whose vertex is at the intersection of the three planes

$$(\alpha_1, \beta_1, \gamma_1, \delta_1),\quad (\alpha_2, \beta_2, \gamma_2, \delta_2),\quad \text{and}\quad (\alpha_3, \beta_3, \gamma_3, \delta_3).$$

Hence determine the number of tangent lines of different classes which can be drawn to the surface from any point.

9779. Prove the following identities

$$(1)\quad \begin{vmatrix} 0, & 1, & 1, & 1, & \ldots & 1 \\ 1, & 0, & x_1+x_2, & x_1+x_3 & \ldots & x_1+x_{n+1} \\ 1, & x_1+x_2, & 0, & x_2+x_3 & \ldots & x_2+x_{n+1} \\ 1, & x_1+x_3, & x_2+x_3, & 0 & \ldots & x_3+x_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & x_1+x_{n+1}, & x_2+x_{n+1}, & x_3+x_{n+1} & \ldots & 0 \end{vmatrix}$$
$$= -(-2)^n x_1 x_2 \ldots x_{n+1} \left\{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_{n+1}}\right\},$$

$$\begin{vmatrix} 0, & x_1+x_2, & x_1+x_3 & \dots & x_1+x_{n+1} \\ x_1+x_2, & 0, & x_2+x_3 & \dots & x_2+x_{n+1} \\ x_1+x_3, & x_2+x_3, & 0 & \dots & x_3+x_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ x_1+x_{n+1}, & x_2+x_{n+1}, & x_3+x_{n+1} & \dots & 0 \end{vmatrix}$$

$$= \frac{1}{2}(-2)^n x_1 x_2 \dots x_{n+1} \left\{ (x_1+x_2+\dots x_{n+1}) \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots \frac{1}{x_{n+1}} \right) - (n-1)^2 \right\}.$$

$$(3) \quad \begin{vmatrix} 0, & 1, & 1 & \dots & \text{to } n \text{ columns} \\ 1, & 0, & 1 & \dots & ,, \\ 1, & 1, & 0 & \dots & ,, \\ \dots & \dots & \dots & & ,, \end{vmatrix} = (-1)^{n-1}(n-1),$$

the last to be proved independently, and then shown to be consistent with the first two.

9780. In space of n dimensions, the spherical loci (hyper-spheres) described about the $n+2$ simplicissima (see Question 8242), each of which is bounded by $n+1$ out of $n+2$ given linear loci, all pass through the same point, when n is even; but not, in general, when n is odd. In space of two dimensions this is MICQUEL's Theorem.

9781. If $ax+by+cz+dw=0$, and $a'x+b'y+c'z+d'w=0$, where $a, b, c, d, a', b', c', d'$ are functions of a parameter θ, be the equations to a line, this line will generate a ruled surface of order $m+n$, where m and n are the orders of the two given equations as functions of θ. Especially examine the case when $m=n=1$, and show that a second set of lines exists in this case.

9782. If P_{nr} denote the coefficient of x^r in the expansion of $(1+x)^n$, &c., C_{nr} denote the number of combinations of n things taken r together; form the equations of differences which determine P_{nr}, and C_{nr}, and hence show that these are equal.

9783. If ABC be a triangle, in which AC > AB, AD the perpendicular from A upon BC, and if DC′ be taken upon CD produced, so that DC′ = CD; the circle through A, B, C′ will also pass through the orthocentre of ABC, and will be equal to the circumcircle of that triangle, which will pass through the orthocentre of ABC′.

9784. If the sums of the squares of the opposite edges of a tetrahedron be equal to one another, show that the nine-point circles inscribed in the triangular faces are all sections of the same sphere; show also that this is the condition that the perpendiculars from the vertices on the opposite faces should meet in a point. Also show in space of n dimensions, that if (rs) denote the edge joining the rth and sth vertices of a simplicissimum (Question 8242), and $(r.s)^2 = A_r + A_s$ (where $A_1, A_2 \dots A_{n+1}$ are $n+1$ areal magnitudes) for all values of r and s, the perpendiculars from the vertices upon the opposite faces will all meet in a point, and the nine-point circles of all the triangles formed by joining the vertices are sections of the same spheric (hyper-sphere).

9785. If $m \nless m'$, there are in general $(m-1)^{2n}(n+1)$ points which have the same linear polar with respect to each of two loci, of orders m and m', in space of n dimensions. Hence deduce the conditions that the loci may touch.

9786. If a circle cut the sides of the triangle of reference at the feet of concurrent lines from the vertices, the line joining the isogonal conjugates of the points of intersection passes through the centroid. Enunciate the corresponding proposition in the Geometry of Higher Space.

9787. If the sums of the opposite edges of a tetrahedron be equal to one another, show that the circles inscribed in the triangular faces are all sections of the same sphere. Also show, in space of n dimensions, that if for all values of r and s [$(r.s)$ denoting the edge joining the rth and sth vertices of a simplicissimum (Question 8242)], $(r.s) = d_r + d_s$, where d_1, d_2, &c., d_{n+1} are $n+1$ linear magnitudes, the circles inscribed in the triangles formed by joining the vertices are all sections of the same spheric (hyper-sphere).

9788. In space of n dimensions, the Jacobian of $n+1$ quadratic loci (which is a locus of the $n+1^{\text{th}}$ order) is the locus of the points which are conjugate with respect to each of the loci, or the locus of the points whose first polars with respect to all the quadratic loci meet in a point.

9789. Show that

$$(\beta-\gamma)(\gamma-\delta)(\delta-\beta)(x-\alpha)^2+(\delta-\gamma)(\gamma-\alpha)(\alpha-\delta)(x-\beta)^2$$
$$+(\delta-\alpha)(\alpha-\beta)(\beta-\delta)(x-\gamma)^2+(\beta-\alpha)(\gamma-\beta)(\alpha-\gamma)(x-\delta)^2$$

vanishes identically, and hence deduce that the sextic covariant J of the binary quantic $\quad (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$$\equiv \{(\alpha-\beta)(x-\gamma)(x-\delta)-(\gamma-\delta)(x-\alpha)(x-\beta)\}\{(\alpha-\gamma)(x-\beta)(x-\delta)$$
$$-(\delta-\beta)(x-\alpha)(x-\gamma)\}\times\{(\alpha-\delta)(x-\beta)(x-\gamma)-(\beta-\gamma)(x-\alpha)(x-\delta)\}$$
$$\equiv \{(\alpha-\beta)(x-\gamma)(x-\delta)+(\gamma-\delta)(x-\alpha)(x-\beta)\}\{(\alpha-\gamma)(x-\beta)(x-\delta)$$
$$+(\delta-\beta)(x-\alpha)(x-\gamma)\}\times\{(\alpha-\delta)(x-\beta)(x-\gamma)+(\beta-\gamma)(x-\alpha)(x-\delta)\};$$

and confirm this by showing that

$$(\alpha-\beta)(x-\gamma)(x-\delta)-(\gamma-\delta)(x-\alpha)(x-\beta)$$
$$\equiv(\alpha-\gamma)(x-\beta)(x-\delta)+(\delta-\beta)(x-\alpha)(x-\gamma).$$

Also explain the geometrical relation between the points determined by the roots of the covariant, and those determined by the roots of the quantic.

9790. If u be a rational and integral symmetrical function of $x_1, x_2 \dots x_n$, show that $\quad x_r^p \frac{du}{dx_r} - x_s^p \frac{du}{dx_s}$ and $x_s^p \frac{du}{dx_r} - x_r^p \frac{du}{dx_s}$

are divisible by $x_r - x_s$ for all positive integral values of p, r, and s.

9791. Show that the secondary form of MACLAURIN's Theorem, given in BOOLE's *Finite Differences*, p. 23, Ed. 1., viz.,

$$\phi(t) = \phi(0) + \phi\left(\frac{d}{d0}\right)0.t + \phi^2\left(\frac{d}{d0}\right)0^2.\frac{t^2}{1.2} + \&c.,$$

leads at once to the equation $\quad \dfrac{t}{e^t-1} = \dfrac{\log(1+\Delta)}{\Delta}.e^{0.t},$

where the Δ's only act on the zeros. Also from this form deduce

$$(e^x-1)^n = \sum_{m=0}^{m=\infty}(\Delta^n.0^m)\frac{x^m}{m!}.$$

9792. **Show that, if** $(p_0 x^n - p_1 x^{n-1} + p_2 x^{n-2} \dots)^m$

$$= q_0 x^{mn} - q_1 x^{mn-1} + q_2 x^{mn-2} - q_3 x^{mn-3} + \&c.,$$

$$q_r = \frac{1}{r!}\left(p_1 \frac{d}{dp_0} + 2p_2 \frac{d}{dp_1} + 3p_3 \frac{d}{dp_1} + \&c.\right) . (p_0)^m,$$

and that, if $m = 2$, $q_r = p_r p_0 + p_{r-1} . p_1 + p_{r-2} . p_2 \dots + p_0 p_r$.

Also deduce the expansion of $\phi(fx)$ in powers of x, where $\phi(x)$ is an integral and rational function of x, and

$$f(x = p_0 x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \&c.$$

9793. If $U = 0$ be the equation to an m-ic locus in space of n dimensions, referred to simplicissimum content coordinates, and

$$S \equiv \lambda\mu (1 . 2)^2 + \mu\nu (2 . 3)^2 + \dots,$$

where (1 . 2), &c. are the edges of the simplicissimum of reference, the equations to the normal to U at the point $(\lambda\mu \dots)$ are

$$\left\| \begin{matrix} \dfrac{dS'}{d\lambda'} - \dfrac{dS}{d\lambda}, & \dfrac{dS'}{d\mu'} - \dfrac{dS}{d\mu} & \dots \\ \dfrac{dU}{d\lambda}, & \dfrac{dU}{d\mu} & \dots \\ 1, & 1 & \dots \end{matrix} \right\| = 0,$$

where $\lambda'\mu'$, &c. are the current coordinates. Hence show that m^n normals can, in general, be drawn from any point to $U = 0$.

9794. If (1 . 2), &c. denote the edges of the tetrahedron of reference, and λ, μ, ν, π be tetrahedral coordinates, show that the circle at infinity is represented by the equations

$$\lambda + \mu + \nu + \pi = 0, \text{ and } \lambda\mu (1 . 2)^2 + \mu\nu (2 . 3)^2 + \dots = 0;$$

and that in space of n dimensions, if (1 . 2), &c. be the edges of the simplicissimum of reference, and λ, μ, $\nu \dots$ content coordinates (see Question 8242), the hypersphere at infinity (in space of n dimensions) is represented by $\lambda + \mu + \nu + \dots = 0$, and $\lambda\mu (1 . 2)^2 + \mu\nu (2 . 3)^2 + \dots = 0$. Hence determine the conditions that an equation of the second degree in tetrahedral or higher content coordinates may represent a sphere or hypersphere.

9795. (Suggested by Question 5420.) If $S_{i.j}$ denote the coefficient of t_j in the developed product of $(1 + t)(1 + 2t) \dots (1 + it)$, show that

$$S_{i.j} = S_{i-1.j} + iS_{i-2.j-1} + i(i-1) S_{i-3.j-2} + \dots$$
$$+ i(i-1) \dots (i-j+1) S_{i-j-1.0}.$$

and that the product itself is the coefficient of x^{i+1} in the expansion of $(1 - tx)^{-1/t}$ multiplied by $(i + 1)!$

9796. Show that the transformation from rectangular to areal coordinates, or *vice versâ*, may be effected by substitution from the equations

$$(\lambda + \mu + \nu) x = \lambda x_1 + \mu x_2 + \nu x_3, \quad (\lambda + \mu + \nu) y = \lambda y_1 + \mu y_2 + \nu y_3,$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the vertices of the triangle of reference.

And similarly, that the rectangular and tetrahedral coordinates of a point in space of three dimensions are connected by the equations

$$(\lambda+\mu+\nu+\pi)\,x = \lambda x_1+\mu x_2+\nu x_3+\pi x_4,$$
$$(\lambda+\mu+\nu+\pi)\,y = \lambda y_1+\mu y_2+\nu y_3+\pi y_4,$$
and
$$(\lambda+\mu+\nu+\pi)\,z = \lambda z_1+\mu z_2+\nu z_3+\pi z_4,$$

or more generally that, in space of n dimensions, the connection between orthogonal and simplicissimum content coordinates (see Question 8242) is given by the equations

$$(\lambda+\mu+\nu\,\dots\,+\tau)\,x = \lambda x_1+\mu x_2+\,\dots\,+\tau x_{n+1},$$
$$(\lambda+\mu+\nu\,\dots\,+\tau)\,y = \lambda y_1+\mu y_2+\,\dots\,+\tau y_{n+1},$$
&c. &c.

9797. If $P\dfrac{d^2y}{dx^2}+2Q\dfrac{dy}{dx}+Ry = X$, where P, Q, R, X are functions of x only, and are subject to the condition $\dfrac{d}{dx}\left(\dfrac{P}{Q}\right)+\dfrac{PR}{Q^2}-1=0$,

show that
$$y = \epsilon^{-\int Q/P\,dx}\iint \frac{X}{P}\,\epsilon^{\int Q/P\,dx}\,dx^2.$$

9798. A quadratic locus in space of n dimensions, has n principal axes (*i.e.*, axes which are at right angles to the linear loci which bisect chords parallel to the axis).

9799. Deduce the solution of $\dfrac{d^2z}{dx^2}-a^2\dfrac{d^2z}{dy^2}=0$ from the expansion of $\phi(x, y)$ in ascending powers of x and y.

9800. If a, b, A be given in a spherical triangle, deduce the conditions that the triangle should be impossible, unique, or ambiguous, from the discussion of the equation

$$\cos a = \cos b\cos c+\sin b\sin c\cos A,$$

where there are two triangles; show that, c and c' being the third sides,

$$\tan\tfrac{1}{2}(c+c') = \tan b\cos A,$$

and confirm this by the case when the radius of the sphere is infinite.

9801. If P_r denote the Legendre's coefficient of the r^{th} order of $\frac{1}{2}(k+1/k)$, show that

$$\int_0^x \frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}} = x+P_1\frac{kx^3}{3}+P_2\frac{k^2x^5}{5}+\dots+P_r\frac{k^rx^{2r+1}}{2r+1}+\&c.$$

9802. If there be two series of functions of x, P_0, P_1, $P_2\dots$, and Q_0, Q_1, $Q_2\dots$, and one of operations, R_0. R_1, R_2, &c., each of which gives a result independent of x: then, if $R_m.P_n.Q_p=0$, whenever m, n, and p are not all equal, but not when they are, any function $f(x)$ may be developed in the forms ΣA_nP_n, or ΣB_nQ_n. Apply this to some known expansions.

9803. Show that for any proper cubic the Cayleyan of the Hessian is the Hessian of the Cayleyan; and that the discriminant of the polar conic of any line vanishes doubly when the line touches the Cayleyan.

9804. Show that the nodes on the locus of a point rigidly connected with the middle bar of a three bar system lie upon the fixed centrode.

9805. If PBC be a small circle of a sphere, B and C fixed points, and P any other point upon it, then, if the arcs BC, PC, and PB be bisected in D, E, and F respectively, and if the arc DE meet BP in B′ and B″, and DF meet CP in C′ and C″; show that (1), for all positions of P, B′, C′, B″, C″ lie on the same great circle, that of which O the pole of the small circle is the pole, and that $B'C' = B''C'' = \pi - B'C'' = \pi - B''C'$; (2) that these arcs are of constant length for every position of P on one side of BC; (3) that the values of those arcs corresponding to positions of P on opposite sides of BC are supplementary; (4) that the points B′, B″, and C′, C″ are the poles of the arcs OF and OE which bisect the sides BP and CP of the triangle BPC at right angles; (5) that the six points in which the sides of the triangle DEF meet the corresponding sides of the triangle PBC lie on the great circle of which O is the pole.

9806. If ABC be an acute-angled triangle, α, β, γ the circular measures of its angles; show that P being a random point in the triangle, the chance that the angle BPC is obtuse, is

$$\frac{\sin\alpha}{2\sin\beta\sin\gamma}\left\{\sin 2\beta + \sin 2\gamma + \pi - 2\alpha\right\}.$$

If the angle at B (β) be obtuse, the chance is

$$\frac{\sin\alpha}{2\sin\beta\sin\gamma}\left\{2\gamma + \sin 2\gamma\right\}.$$

9807. The perpendiculars from the vertices of a triangle upon the central axis (the line which passes through the circumcentre, the orthocentre, the nine-point centre, and the centroid) are proportional to

$$\cos A\sin(B-C),\quad \cos B\sin(C-A),\ \text{and}\ \cos C\sin(A-B),$$

those on one side of the line being reckoned positive, and those on the other negative.

9808. The mean value of the pedal triangle of a random point in a triangle ABC is $\frac{1}{4}R^2(1+\cos A\cos B\cos C)$, where A, B, and C are the angles of the triangle, and R the circumradius.

9809. Given forces act along the sides of a triangle, in the same sense. The value of the mean sum of their moments about a random point in the triangle is the mean of their moments about the vertices of the triangle. If the forces be such that, if applied at a point, there would be equilibrium, the sum of the moments is the same for all points.

9810. A, B, C are fixed points, P another point, construct the resultant of forces acting along PA, PB, PC, when they are proportional to

$$(1)\ PA^2,\ PB^2,\ \text{and}\ PC^2;\quad (2)\ \frac{1}{PA},\ \frac{1}{PB},\ \text{and}\ \frac{1}{PC};$$

$$(3)\ \frac{1}{PA^2},\ \frac{1}{PB^2},\ \text{and}\ \frac{1}{PC^2}.$$

9811. If masses P, Q, and R be placed at the vertices A, B, and C respectively of the triangle of reference, show that the trilinear equations to the principal axes at the C. of G. of the masses will be

$$lx + my + nz = 0,\ \text{and}\ l'x + m'y + n'z = 0,$$

where $l : m : n$ and $l' : m' : n'$ are determined by the equations

$$P\frac{l}{a} + Q\frac{m}{b} + R\frac{n}{c} = 0, \quad P\frac{l'}{a} + Q\frac{m'}{b} + R\frac{n'}{c} = 0,$$

$$P\frac{ll'}{a^2} + Q\frac{mm'}{b^2} + R\frac{nn'}{c^2} = 0, \text{ and}$$

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0,$$

and that at any point the principal axes are conjugate (*i.e.*, each passes through the pole of the other) with respect to the conics

$$\frac{Pl^2}{a^2} + \frac{Qm^2}{b^2} + \frac{Rn^2}{c^2} = 0$$

and $\quad l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0$

(the circular points at infinity).

9812. If P_n denote $\int_0^x \frac{x^{2n}}{(1-x^2)^{n+1}}$, and Q_n, $\int_0^{kx} \frac{x^{2n}}{(1-x^2)^{n+1}}$;

show that

$$P_n = \frac{1}{2n}\frac{x^{2n-1}}{(1-x^2)^n} - \frac{2n-1}{2n}P_{n-1},$$

$$Q_n = \frac{1}{2n}\frac{k^{2n-1}x^{2n-1}}{(1-k^2x^2)^n} - \frac{2n-1}{2n}Q_{n-1};$$

and that

$$\int_0^x \frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}}$$

$$= P_0 - \frac{1}{1}\left(\frac{k'^2}{2}\right)P_1 + \frac{1.3}{1.2}\left(\frac{k'^2}{2}\right)^2 P_2 - \frac{1.3.5}{1.2.3}\left(\frac{k'^2}{2}\right)^3 P_3 + \&c.$$

$$= \frac{1}{k}\left\{Q_0 + \frac{1}{1}\left(\frac{k'^2}{2k^2}\right)Q_1 + \frac{1.3}{1.2}\left(\frac{k'^2}{2k^2}\right)^2 Q_2 + \frac{1.3.5}{1.2.3}\left(\frac{k'^2}{2k^2}\right)^3 Q_3 + \&c.\right\}.$$

9813. Show that

$$\int\frac{x^{2n}}{(1-x^2)^{\frac{1}{2}}}dx = -\frac{x^{2n-1}}{2n}(1-x^2)^{\frac{1}{2}} + \frac{2n-1}{2n}\int\frac{x^{2n-2}}{(1-x^2)^{\frac{1}{2}}}dx,$$

and therefore

$$\int_0^1 \frac{x^{2n}}{(1-x^2)^{\frac{1}{2}}}dx = \frac{2n-1}{2n}\int_0^1 \frac{x^{2n-2}}{(1-x^2)^{\frac{1}{2}}}dx.$$

And hence show that

$$\text{(i.)}\ \pi = \frac{2^{2n+1}(n!)^2}{2n!}\left\{\frac{1}{2n+1} + \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{2n+3} + \frac{1.3}{1.2}\frac{1}{2^2}\cdot\frac{1}{2n+5} + \frac{1.3.5}{1.2.3}\cdot\frac{1}{2^3}\cdot\frac{1}{2n+7} + \&c.\right\};$$

$$\text{(ii.)}\ \int_0^1 \frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}} = \frac{\pi}{2}\left\{1 + \frac{1^2}{1^2}\cdot\left(\frac{k}{2}\right)^2 + \frac{1^2.3^2}{1^2.2^2}\cdot\left(\frac{k}{2}\right)^4 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\left(\frac{k}{2}\right)^6 + \&c.\right\}.$$

9814. **Show that, for all integer values of n,**

$$\frac{1}{2n+1} - \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{2n+3} + \frac{1\,.\,3}{1\,.\,2}\cdot\frac{1}{2^2}\cdot\frac{1}{2n+5} - \frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\cdot\frac{1}{2^3}\cdot\frac{1}{2n+7} + \&c.$$

$$= \frac{1}{2^n}\left\{1 - \frac{2n-1}{(2n-1)} + \frac{(2n-1)(2n-3)}{2\,(n-1)\,.\,2\,(n-2)} - \&c.\right\}$$

$$+ (-1)^n \frac{1}{2^{2n+1}}\cdot\frac{(2n\,!)}{(n\,!)^2}\log\left(1+\sqrt{2}\right).$$

9815. If dashes above the line denote differentiation with respect to x, and dashes below with respect to y, and ϕ and ψ stand for any functions of x and y; show that the equation

$$(\phi'\psi_{,} - \psi'\phi_{,})\frac{d^2y}{dx^2} + \left\{\phi_{,}\psi_{,,} - \psi_{,}\phi_{,,} + a\phi_{,}^{2}\psi_{,} + b\psi\phi_{,}^{3}\right\}\left(\frac{dy}{dx}\right)^3$$

$$+ \left\{2\,(\phi_{,}\psi_{,}' - \psi_{,}\phi_{,}') + \phi'\psi_{,,} - \psi'\phi_{,,} + a\,(\psi'\phi_{,}^{2} + 2\phi'\phi_{,}\psi_{,}) + 3b\psi\phi'\phi_{,}^{2}\right\}\left(\frac{dy}{dx}\right)^2$$

$$+ \left\{2\,(\phi'\psi_{,}' - \psi'\phi_{,}') + \phi_{,}\psi'' - \psi_{,}\phi'' + a\,(\phi'^2\psi_{,} + 2\phi'\phi_{,}\psi') + 3b\psi\phi'^2\phi_{,}\right\}\frac{dy}{dx}$$

$$+ \phi'\psi'' - \phi''\psi' + a\phi'^2\psi' + b\psi\phi'^3 = 0$$

may always be transformed into a linear differential equation with constant coefficients.

9816. If the perpendiculars PA″, PB″, PC″ from any point P on the conic $\lambda yz + \mu zx + \nu xy - \mathrm{K}\,(x\sin \mathrm{A} + y\sin \mathrm{B} + z\sin \mathrm{C})^2 = 0$ be produced to A′, B′, and C′ respectively, and if

$$\mathrm{PA'} = \frac{\sin \mathrm{A}}{\lambda}\,.\,\mathrm{PA''},\quad \mathrm{PB'} = \frac{\sin \mathrm{B}}{\mu}\,.\,\mathrm{PB''},\quad \mathrm{PC'} = \frac{\sin \mathrm{C}}{\nu}\,.\,\mathrm{PC''},$$

show that the area of the triangle A′B′C′ is constant.

9817. If A′, B′, C′ be the reflections of any point P, on the circumcircle of the triangle ABC, with respect to the sides; show, by Euclid, that A′, B′, C′ lie in a straight line which passes through the orthocentre of ABC. Hence deduce the theorem (Quest. 2145) that "the feet of the perpendiculars let fall on the sides of a triangle from any point on the circumscribing circle lie in a straight line. Show that this straight line is equidistant from the point and from the centre of perpendiculars of the triangle."

9818. If ABC be a triangle, A′, B′, C′ the feet of the perpendiculars from any point upon the sides, and A″, B″, C″ points in PA′, PB′, PC′ (produced if necessary) such that PA″ $= f$.PA′, PB″ $= g$.PB′, PC″ $= h$.PC′, respectively; show that, when A″, B″, C″ lie in a straight line, the locus of P is a conic circumscribed to ABC; and that, when the area of A″B″C″ is constant, the locus of P is a conic having double contact with first at infinity. Also, when the conic is given, determine $f : g : h$.

9819. If the sides AB and AC of a spherical triangle ABC be divided in F and E respectively, so that sin AF : sin BF :: sin AE : sin CE, the great circle FE will cut the great circle BC in a point Q such that BQ + CQ $= \pi$, and the great circles through all such divisions meet in the same points, and conversely.

9820. Prove the following—(i.) If *abc* and A′B′C′ be the pedal triangles of the circumcentre O of the triangle ABC and of any other point P, $A'B'C' = \frac{1}{4}\Delta ABC \times (R^2 \sim OP^2)/R^2$, where R is the circumradius. (ii.) If O and K be the centres of two circles whose radii are R and r, P any point on the second circle, and PL the perpendicular from P to the radical axis of the circles, $2OK \,.\, PL = R^2 \sim OP^2$. (iii.) The area of the pedal triangle of any point P on a circle, the centre of which is K, with respect to a triangle ABC, of which O is the circumcentre and R the circumradius, is $\frac{1}{2}\Delta ABC\,(OK \,.\, PL)/R^2$, where PL is the perpendicular from P upon the radical axis of the two circles.

9821. Show that, if a point be taken at random in the circumscribed circle of a triangle, the mean area of the pedal triangle is $\frac{1}{8}$ of the triangle.

9822. If $y = x^{n-1}\log x$, prove that, when r is not $> n$,

$$\frac{d^r y}{dx^r} = (n-1)(n-2)\dots(n-r)\,x^{n-r-1}\left\{\log x + \left(\frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{n-r}\right)\right\};$$

and hence show that

$$\begin{aligned}(1+x)^{n-1}\log(1+x) &= x + \frac{(n-1)(n-2)}{1\,.\,2}\left\{\frac{1}{n-1} + \frac{1}{n-2}\right\}x^2 \\ &+ \frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}\left\{\frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3}\right\}x^3 + \dots \\ &+ \left\{\frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3} + \dots + \frac{1}{1}\right\}x^{n-1} \\ &+ \frac{x^n}{n}\left\{1 - \frac{1\,.\,x}{n+1} + \frac{1\,.\,2\,.\,x^2}{(n+1)(n+2)} - \frac{1\,.\,2\,.\,3\,x^3}{(n+1)(n+2)(n+3)} + \&c.\right\}.\end{aligned}$$

9823. Prove (1)

$$\int\frac{d\theta}{(a\cos^2\theta + b\sin^2\theta)^2} = \frac{a+b}{2a^{\frac{3}{2}}b^{\frac{3}{2}}}\tan^{-1}\left\{\left(\frac{b}{a}\right)^{\frac{1}{2}}.\tan\theta\right\} - \frac{a-b}{2ab}\,\frac{\tan\theta}{a+b\tan^2\theta};$$

(2) that $X \equiv a + bx^n$,

$$\int x^{m-1}\log x\,.\,X^p\,dx = \frac{x^m(m\log x - 1)}{m^2}X^p - \frac{bnp}{m^2}\int x^{m+n-1}(m\log x - 1)X^{p-1}dx;$$

(3) that $$\int_0^{\frac{1}{2}\pi}\cos^m\phi\sin^n\phi\,.\,d\phi = \frac{1}{2}\,\frac{\Gamma\left[\frac{1}{2}(m+1)\right]\Gamma\left[\frac{1}{2}(n+1)\right]}{\Gamma\left[\frac{1}{2}(m+n)+1\right]}.$$

9824. If the diagonals of a quadrilateral are at right angles, the sums of the squares of the opposite sides are equal. Hence, if a, b, c, d be the sides in order, a, c, b, d will be the sides in order of a quadrilateral in a circle the area of which is $\frac{1}{2}(ac + bd)$.

9825. If particles be projected along lines meeting in a point, with velocities proportional to the projections of the same vertical line upon each line; at any time the particles will lie upon the sphere described upon the space traversed by the vertical particle as diameter. Hence find the line of quickest transit when an inelastic particle is dropped from O to A (a vertical distance h), and after impact upon an inelastic line AB

(the line to be determined) proceeds along it to B, a point on a given circle in a vertical plane through OA.

9826. If particles be projected from a point A, with the same velocity V, along lines meeting at A; at any time (t) they will all lie upon the surface generated by the revolution, about the vertical line through A, of the bicircular quartic, which is the inverse about A of a conic whose focus is at A, its directrix horizontal at a distance $\frac{2k^2}{gt^2}$ upwards from A $\left(\text{where } k \text{ is the radius of inversion, and its eccentricity } \frac{Vt}{k^2}\right)$. The conics corresponding to different values of t envelop a circle of which the highest point is at A and the radius $\frac{gk^2}{V^2}$.

9827. If $U = 0$ be the homogeneous equation to a curve of order n, and
$$\Delta \equiv x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1};$$
show that the discriminant of
$$\lambda^n U_1 + \lambda^{n-1}\mu\Delta U_1 + \tfrac{1}{2}\lambda^{n-2}\mu^2\Delta^2 U_1 + \&\text{c.},$$
only differs by a factor from the result of substituting $\left\|\begin{matrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right\|$ for α, β, and γ in the tangential equation to the curve.

9828. If $U = 0$ be the homogeneous equation to a surface of order n,
and
$$\Delta \equiv x_2 \frac{d}{dx_1} + y_2 \frac{d}{dy_1} + z_2 \frac{d}{dz_1} + w_2 \frac{d}{dw_1},$$
and
$$\Delta' \equiv x_3 \frac{d}{dx_1} + y_3 \frac{d}{dy_1} + z_3 \frac{d}{dz_1} + w_3 \frac{d}{dw_1};$$
show that the discriminant of
$$\lambda^n U_1 + \lambda^{n-1}(\mu\Delta + \nu\Delta')\, U_1 + \tfrac{1}{2}\lambda^{n-2}(\mu\Delta + \nu\Delta')^2\, U_1 + \&\text{c.}$$
can only differ by a factor from the result of substituting $\left\|\begin{matrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \end{matrix}\right\|$ for α, β, γ, and δ in the tangential equation to the surface.

Show that this may be extended to higher space.

APPENDIX III.

UNSOLVED QUESTIONS.

1087. (The EDITOR.)—ABCD is a conic whose centre is O. If the radii vectores OA, OB, OC, OD represent in magnitude and direction four forces, show that the direction of the resultant passes through the centre of a second conic which is parallel to the first, and passes through the points A, B, C, D.

1196. (The EDITOR.)—Given the vertical angle and *one* of the containing sides, construct the triangle, when the ratio of the base to the sum of the other side and a given line is given, or a minimum.

1207. (ALPHA.)—To determine the position of a rock (R), the angles subtended at it by the distances between three headlands (A, B, C) were observed, viz.,

$$BRC = 151^\circ\, 2'\, 42'', \quad CRA = 133^\circ\, 25'\, 57'', \quad ARB = 75^\circ\, 31'\, 21'';$$

and it was known, from a previous survey, that AB = 8883, BC = 9870, and CA = 10857 yards. Find the distance of the rock from each of the headlands.

1228. (ALPHA.)—A messenger M starts from A towards B (distance a) at a rate of v miles per hour; but before he arrives at B, a shower of rain commences at A and at all places occupying a certain distance z towards, but not reaching beyond, B, and moves at the rate of u miles an hour towards A; if M be caught in this shower, he will be obliged to stop until it is over; he is also to receive for his errand a number of shillings inversely proportional to the time occupied in it, at the rate of n shillings for one hour. Supposing the distance z to be unknown, as also the time at which the shower commenced, but all events to be equally probable, show that the value of M's expectation is, in shillings,

$$\frac{nv}{a}\left\{\frac{1}{z}-\frac{u}{v}+\frac{u(u+v)}{v^2}\log\frac{u+v}{u}\right\}.$$

1274. (The EDITOR.)—If an indefinite number of parallel equidistant lines is drawn on a plane, and a regular polygon, the diameter of whose circumscribed circle is less than the distance between consecutive parallels, is thrown at random on the plane; prove that the probability that the polygon will fall on one of the lines is l/l', where l is the perimeter of the polygon, and l' the circumference of the greatest circle that can be placed between the parallels.

1913. (The late Rev. R. H. WRIGHT, M.A.)—Find the condition in order that a straight line passing through an angular point of the triangle

of reference shall be a normal to the conic whose equation is

$$(l\alpha)^{\frac{1}{3}} + (m\beta)^{\frac{1}{3}} + (n\gamma)^{\frac{1}{3}} = 0.$$

1916. (Sir R. BALL, LL.D., F.R.S.) — Show that the equation of squares of differences of the biquadratic $(a, b, c, d, e)(x, 1)^4 = 0$ has for its discriminant (H being $= b^2 - ac$, &c., as in Quest. 1876)

$$(27J^2 - I^3)^2 (4H^3 - a^2IH - a^3J)^2 (55296H^3J + 2304aH^2I^2 - 16632a^2HIJ - 625a^3I^3 - 9261a^2J^2)^2.$$

1919. (The late Professor TOWNSEND, F.R.S.)—If a system of quadrics touch a common system of eight, seven, or six planes, their director spheres (that is, the spheres which are the loci of the intersections of their rectangular triads of tangent planes) have a common radical plane, axis, or centre.

Prove the three general properties involved in this statement; and show from them, respectively, that—

(1) The director spheres of all quadrics passing through the four sides of any skew quadrilateral have a common radical plane with the two spheres of which the two diagonals are diameters.

(2) The director spheres of all quadrics passing through a common line and touching four common planes, have a common radical axis with the four spheres of which the four connectors of the intersection of three planes with that of the line and fourth are diameters.

(3) The diameter spheres of all quadrics touching six common planes, have a common radical centre with those of the fifteen quadrics determined by the fifteen different triads of intersections of the planes taken in pairs.

1928. (N'IMPORTE.) — Given the four cones $-cy^2 + bz^2 - fw^2 = 0$, $cx^2 - az^2 - gw^2 = 0$, $-bx^2 + ay^2 - hw^2 = 0$, $fx^2 + gy^2 + hz^2 = 0$, and the four conics which are the sections of these by the planes $x = 0$, $y = 0$, $z = 0$, $w = 0$, respectively; prove that (1) any line touching three of the four cones touches the fourth cone, and (2) any line meeting three of the four conics meets the fourth conic.

1935. (N'IMPORTE.)—(1) Three points are marked at random on a given straight line; find the chance that, of the four parts into which it is thus divided, any three will be together greater than the fourth.

(2) Again, a rod of given length has a piece cut off at random; from the remainder a piece is again cut off at random, and the piece then left is divided into four parts at random: find the chance that any three of the parts will be together greater than the fourth.

1989. (Professor CREMONA.)—On donne un tétraèdre *abcd*. Considérons tous les cones quadriques S qui ont leur sommet en *c* et sont tangents aux plans *cad*, *cbd* le long des droites *ca*, *cb*; et tous les cones quadriques S′ qui ont leur sommet en *a* et touchent les plans *adc*, *abc* suivant les droites *ad*, *ab*. Un cone S et un cone S′ étant choisis arbitrairement, se coupent suivant une courbe gauche C du 4^e ordre; et toutes ces courbes C ont un rebroussement en *a* et sont osculées en *b* par un même plan stationnaire *cbd*, etc. (voir *Comptes Rendus* 17 mars 1862, p. 604). Démontrer géométriquement les propriétés qui suivent:—(1) Si d'un point quelconque *p* de l'espace on mène les plans osculateurs à toutes les courbes C, les points de contact formeront une surface P de 3^e ordre et 4^e classe, qui passe par le point donné *p* et par les six arêtes du tétraèdre *abcd*; (2) Les plans osculateurs des courbes C, aux points où celles-ci sont

coupées par un plan quelconque donné π, enveloppent une surface Π de 3e classe et 4e ordre qui touche le plan π et passe par les six arêtes du tétraèdre ; (3) Il y a une corrélation de figures, dans laquelle à un point p, donné arbitrairement dans l'espace, correspond le plan π qui oscule en p la courbe C qui passe par ce point ; et, *vice versâ*, à un plan donné π correspond le point p de contact entre ce plan et la courbe C qui est osculée par ce même plan. Si le point p décrit un plan π', le plan π enveloppe la surface Π' ; si le plan π tourne autour d'un point fixe p', le lieu du point p est la surface P′, etc., etc.

1999. (R. Tucker, M.A.)—(1) P is a given point on the side of a triangle, and Q another given point in the same plane: it is required to inscribe in the triangle a maximum triangle having P for a vertex and its base passing through Q. Again (2), P is a given point on a circle, and Q a point in the same plane with it: it is required to inscribe in the circle a maximum triangle having P for its vertex and its base passing through Q. Discuss the several cases fully.

2233. (The late T. Cotterill, M.A.)—1. If two points are given, there are two others in the same plane such that the distances of points taken from each pair vanish. In orthogonal diameters of orthogonal circles, the diameter of each circle cuts the other in such pairs of points.

The products of the distances of a point in the same plane from the pairs are equal, and the cosine of the angle subtended at the point is real.

2. If a triangle is given by a point on a circle and the intersections of a line and the circle, give a construction for the centres of the circles touching the sides of the triangle.

2287. (W. B. Davis, B.A.)—Find what algebraical curves can be expressed by arcs of the circle.

2293. (Professor Whitworth.)—If B, B′ be two real points and O, O′ the circular points at infinity, prove that (1) the fourteen-points conic of the quadrilateral BBOO′ is the rectangular hyperbola whose conjugate axis is BB′ ; (2) the critical circumscribing conic of the same quadrilateral is the circle on BB′ as diameter, and the critical inscribed conic is the ellipse whose foci are B, B′, and whose excentricity is $\frac{1}{2}\sqrt{2}$; (3) these three conics have double contact at the vertices of the hyperbola—the minor vertices of the ellipse ; (4) if EE′ be the third diagonal of the quadrilateral BB′OO′, then the critical circumscribing conic of the quadrilateral EE′OO′ is an imaginary circle, concentric with the circle on BB′, their radii being in the ratio $\sqrt{(-1)} : 1$, and the critical inscribed conic is the *imaginary* ellipse whose real foci are B, B′, and whose excentricity is $\frac{1}{2}\sqrt{2}$; and (5) the critical circumscribing conic of the quadrilateral BB′EE′ is the rectangular hyperbola conjugate to the former one, and the critical inscribed conic is another rectangular hyperbola, similarly situated to the last, and having B, B′ as foci.

2314. (The late T. Cotterill, M.A.)—Taking points in a plane ; prove that (1) the sum of the squares of the distances of a fixed point from the four centres of the circles touching the sides of any triangle inscribed in a fixed circle is invariable, and this holds, if two of the points, or even if all three, coincide on the circle ; (2) more generally, if A, B, C, D be four such centres, and we denote the square of the length of the tangent drawn from a point—to the fixed circle by (M) ; to the circle on the

diameter AB, by (AB), &c.; to the circles circumscribing BCD and copolar to it, by (A) and (α), &c.: then we shall have the following system of identities:—

$$\begin{aligned} 2(\mathrm{M}) &= (\mathrm{BC})+(\mathrm{AD}) = (\mathrm{CA})+(\mathrm{BD}) = (\mathrm{AB})+(\mathrm{CD}) \\ &= (\mathrm{A})+(\alpha) = (\mathrm{B})+(\beta) = (\mathrm{C}+\gamma) = (\mathrm{D})+(\delta) \\ &= \tfrac{1}{2}\left\{(\mathrm{A})+(\mathrm{B})+(\mathrm{C})+(\mathrm{D})\right\} = \tfrac{1}{2}\left\{(\alpha)+(\beta)+(\gamma)+(\delta)\right\}. \end{aligned}$$

Hence $AP^2+BP^2+CP^2+DP^2 =$ a constant is the equation to a circle concentric with (M); and (3) if A, B, C, D are any points in the plane, state the nature of a pair of points, corresponding to the circular points at infinity, such that a similar system of equations must exist between the corresponding conics through the two points.

2357. (The late T. Cotterill, M.A.)—

1. If the determinant $\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix}$ vanish, the equations

$$\alpha\,(bz^2+cy^2-2dyz) = \beta\,(cx^2+az^2-2ezx) = \gamma\,(ay^2+bx^2-2fxy)$$

are satisfied by two conjugate pairs of values of the variables, one pair being independent of the constants α, β, γ. Find the equation to the quadric satisfying the values $y=z=0$; $z=x=0$; $x=y=0$, and the last pair of values; and the linear equations satisfying each conjugate pair.

2. Show that such a system of equations is the analytical representation of the projection of the three circles described on the lines connecting the opposite points of intersection of four lines in a plane as diameters, the circle circumscribing their diagonal triangle, and the line at infinity.

2366. (Professor Burnside, M.A., F.R.S.)—Determine (1) the locus of points such that the polar conics with reference to the curve U shall be equilateral hyperbolas, where

$$\mathrm{U} = \Sigma\,\frac{\sin 2\mathrm{A}}{xyz + x^2(-x\cos\mathrm{A}+y\cos\mathrm{B}+z\cos\mathrm{C})};$$

and show (2) that this locus passes through the vertices of the triangle ABC and through the feet of perpendiculars of the same triangle.

2390. (The late G. C. De Morgan, M.A.)—Prove that

$$\int_{-\infty}^{+\infty} 1/\left\{x\,\phi\,(x-a/x)\right\}dx = 0,$$

a being anything positive; and

$$\int_{-\infty}^{+\infty} 1/\left\{x\,\phi\,(x+a/x)\right\}dx = 2\int_{-\infty}^{+\infty} \phi x/x\,dx,$$

a being infinitely small and positive, and ϕ being such a function that the subject of integration is finite for all finite values of x, however small a may be. In the second case, if $\phi x/x$ be infinite when $x=0$, the integral on the right must be replaced by

$$2\int_{-\infty}^{+\infty} 1/\left[x\left\{\phi x-\phi(-x)\right\}\right]dx.$$

2322. (The late Professor Townsend, F.R.S.)—Express the radius (R) of a circle orthogonal to three others in a plane, in terms of their

three radii (p, q, r) and the three sides (a, b, c) of the triangle determined by their three centres. Investigate the corresponding formula for the radius of the circle orthogonal to three others on the surface of a sphere.

2402. (R. TUCKER, M.A.) — Prove that the locus of a point whose distance from its polar with reference to a given conic is equal to its distance from a given point is a quartic curve, which, when the conic becomes a circle, degenerates into a cubic curve.

2419. (The late T. COTTERILL, M.A.)—1. If AA′, BB′, CC′ are the opposite intersections of a complete quadrilateral, an infinite number of cubics can be drawn through these points and another point D, touching DA, DA′ at A and A′. Amongst these cubics, there are two triads of straight lines and four cubics having respectively a point of inflexion at B, B′, C, C′.

2. The locus of the intersection of tangents at B, B′ is the conic DAA′BB′; and of tangents at C, C′ is the conic DAA′CC′.

Give the reciprocal results when the class cubic degenerates.

2436. (Professor CROFTON, F.R.S.)—If ρ_1, ρ_2, ρ_3, ρ_4 are the distances of a point from four concyclic points 1, 2, 3, 4; and if α, β, γ, δ are the triangles formed by joining 1, 2, 3, 4; then the equation

$$\rho_1\sqrt{(\alpha)}-\rho_2\sqrt{(\beta)}+\rho_3\sqrt{(\gamma)}-\rho_4\sqrt{(\delta)}=0, \text{ or } \rho_1\rho_3\sqrt{(\alpha\gamma)}=\rho_2\rho_4\sqrt{(\beta\delta)},$$

represents two circles cutting each other and the circle 1234 orthogonally.

2439. (The late T. COTTERILL, M.A.)—If a, b, c, d are collinear points as well as x, y, z, t and in the same plane; then of the 16 intersections of ax, ay, bx, by with cz, ct, dz, dt, 8 lie on one conic and 8 on another. The 4 points of intersection of these conics lie on a third conic through $abxy$, and a fourth through $cdzt$, and these conics are respectively harmonics to the lengths (ab, xy) and (cd, zt). Also, the tangents to the four conics at any point of intersection are harmonic.

2440. (S. ROBERTS, M.A.)—Show that the centre of a curve of the nth degree (*i.e.*, the centre of mean distances of the points of contact, if parallel tangents) is the C. M. D. of the poles of the line at infinity.

2442. (The late Professor TOWNSEND, F.R.S.) — Through a given point in a plane draw a line the sum of the squares of whose distances from any number of given points in the plane shall be a maximum, a minimum, or given.

2443. (J. GRIFFITHS, M.A.)—Prove (1) that the Jacobian of the three conics represented by the trilinear equations

$$\begin{aligned}
S &= \sin^2 A\,.\,\alpha^2+\&c.\ -2\sin B\sin C\,.\,\beta\gamma-\&c.=0,\\
S' &= \cos^2 A\,.\,\alpha^2+\&c.\ -2\cos B\cos C\,.\,\beta\gamma-\&c.=0,\\
F &= \sin 2A\,.\,\alpha^2+\&c.\ -2\sin A\,.\,\beta\gamma-\&c.=0,
\end{aligned}$$

breaks up into the three right lines

$$\frac{\beta}{\sin(C-A)}+\frac{\gamma}{\sin(A-B)}=0,\quad \frac{\gamma}{\sin(A-B)}+\frac{\alpha}{\sin(B-C)}=0,$$

$$\frac{\alpha}{\sin(B-C)}+\frac{\beta}{\sin(C-A)}=0.$$

Hence show (2) how to construct geometrically the common self-conjugate triangle of the three conics in question.

2467. (Professor CROFTON, F.R.S.) — A Cartesian oval, having a given focus F, is made to pass through three fixed points on a straight line. Show that the fourth point in which it meets the line is also fixed; and that the locus of the points of contact of its double tangent is a circle with F as centre.

2468. (W. B. DAVIS, B.A.)—Prove that a curve of the fifth order and fifth class has three double points, three points of rebroussement, three double tangents, and three points of inflexion.

2487. (The late Rev. R. H. WRIGHT, M.A.) — If a conic be circumscribed about a triangle ABC, and tangents be drawn at A, B, C, and produced to meet so as to form respectively three triangles having the sides of the triangle ABC for their bases; find forms for the bisectors of the angles of the external triangles, and the equations to their circumscribing circles in trilinear coordinates.

2488. (The late Professor TOWNSEND, F.R.S.)—Apply the method of homographic division to draw the two right lines which intersect four given right lines in spoce.

2496. (The late Dr. BOOTH, F.R.S.)—Let $U \equiv \phi(\xi, v)$ be the tangential equation of any plane curve, t the portion of the tangent between the point of contact and the foot of the perpendicular p from the origin. Then generally

$$\frac{t}{p} = \left\{ \left(\frac{dU}{d\xi}\right) v - \left(\frac{dU}{dv}\right) \xi \right\} \div \left\{ \left(\frac{dU}{dv}\right) \xi + \left(\frac{dU}{dv}\right) v \right\} = \tan i,$$

i being the angle between the perpendicular and the radius vector.

Now a well-known formula for the rectification of plane curves being

$$S = t \pm \int p d\lambda, \text{ in which } \frac{\cos\lambda}{p} = \xi \text{ and } \frac{\sin\lambda}{p} = v,$$

we shall find the rectification of a plane curve of which the tangential equation is given generally as easy as to find the quadrature of that whose projective equation is given. [To apply these principles, take the Caustic of the circle for parallel rays, discussed in the EDITOR'S Note to Question 1509 (Vol. II., p. 21).]

2501. (N'IMPORTE.)—Find the equation whose roots are the differences of the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$.

2508. (Professor CROFTON, F.R.S.)— 1. A point being denoted by (ρ, σ, τ), its tri-polar coordinates or distances from three poles R, S, T, show that all circles represented by $A\rho^2 + B\sigma^2 + C\tau^2 = 0$ are orthogonal to the circle RST. 2. If O be the centre of the circle $A\rho^2 + B\sigma^2 + C\tau^2 = D$, show that A, B, C are proportional to the triangles SOT, TOR, ROS.

2538. (The late T. COTTERILL, M.A.)—Prove that, if the equation to a curve is of the form $x^p y^q z^r = k$ (where $p+q+r = 0$), the order and class are the same, and the singularities are reciprocal; (2) if the variables denote point coordinates, the locus of a point on a tangent to the curve, which with the point of contact is harmonic to the lines $x^2+y^2+2axy=0$, is of the form $P^p . Q^q . R^r = mz^r$, P and Q being linear and R of two dimensions in x and y; and find (3) what is the reciprocal theorem, if the circular points at infinity are the pair of points.

2555. (The late Professor DE MORGAN.)—The following is a theorem of which an elementary proof is desired. It was known before I gave it

in a totally different form in a communication (April, 1867) to the Mathematical Society on the *conic octogram;* and the present form is as distinct from the other two as they are from one another. If I, II, III, IV be the consecutive chord-lines of one tetragon inscribed in a conic, and 1, 2, 3, 4 of another; the eight points of intersection of I with 2 and 4, II with 1 and 3, III with 2 and 4, IV with 1 and 3, lie in one conic section. A proof is especially asked for when the first conic is a pair of straight lines. There is, of course, another set of eight points in another conic, when the pairs 13, 24 are interchanged in the enunciation.

2560. (J. J. Walker, F.R.S.)—Given that either of one pair of impossible roots of the equation $3x^4-16x^3+30x^2+8x+39=0$ gives a real result when substituted for x in $5x^3-18x^2-7x$, it is required to find the four (impossible) roots of the biquadratic.

2564. (The late M. Collins, B.A.)—A being a curve whose equation is given in the usual Cartesian rectangular coordinates, B the evolute of A, and C the evolute of B; required a general differential expression for the radius of curvature of C, on the usual supposition of dx being taken constant, and likewise on the supposition of $dx^2+dy^2\ (=dz^2)$ being taken constant.

2565. (Professor Crofton, F.R.S.) — 1. A convex boundary of any form of length L, encloses an area Ω. If two straight lines are drawn at random to intersect it, the probability of their intersection lying within it is $p=2\pi\Omega\,L^{-2}$.

2. The probability of their intersection lying within any given area ω, which is enclosed within Ω, is $p=2\pi\omega L^{-2}$. [An interesting but much more difficult problem is to find the chance of their intersection lying on a given area ω, *external* to Ω.]

3. If an infinity of random lines are drawn across the given area Ω, their intersections form an assemblage of points covering the plane, the density of which is clearly uniform *within* Ω. Show that at any *external* point P the density varies as $\theta-\sin\theta$, where θ is the angle Ω subtends at P.

4. If Ω be any plane area, enclosed by a convex boundary of length L, and θ be the angle it subtends at any external point P (x, y), prove that

$$\iint(\theta-\sin\theta)\,dx\,dy=\tfrac{1}{2}L^2-\Omega\ ,$$

the integral extending over the whole external surface of the plane.

2578. (Professor Crofton, F.R.S.)—If Δ be the difference between the whole length of a complete hyperbola and that of its asymptotes, and if θ be the angle between the tangents to the curve from any external point (x, y), then $\iint(\theta-\sin\theta)\,dx\,dy=\frac{1}{2}\Delta^2$, the integral extending over the whole surface of the plane outside the hyperbola. [When the two tangents touch the *same* branch, θ is the *exterior* angle which they make.]

2580. (A. W. Panton, B.A.)—1. If F and F′ are the foci of an oval of Cassini, and C its centre; prove the following construction for the second pair of foci. The circle through F or F′ and the two points where any line through C meets either oval (the curve consisting of two distinct ovals) cuts the axis in one of the required points.

2. If S and S′ be the foci thus found, and P any point on the curve, prove that PS . PS′ is proportional to PC².

2602. (Professor CROFTON, F.R.S.)—1. If θ be the angle between the tangents to an ellipse from an external point (x, y), then $\iint \theta \, dx\, dy = \pi\Delta$, the integral extending over the annular space between the curve and any outer similar coaxal ellipse; Δ being the difference of the parts into which that space is divided by any tangent to the inner ellipse.

2. Show that this theorem holds for any two convex boundaries, so related that any tangent to the inner cuts off a constant area from the outer.

3. Show that, if the same integral, with regard to any convex boundary, be extended over the annulus between it and *any* outer convex boundary, $\iint \theta \, dx\, dy = \pi(\Theta - 2\mathrm{A})$, Θ being the area of the annulus, and A the *average area* cut from it by a tangent to the inner boundary (the tangent being supposed to alter by constant angular variations).

2629. (The late Professor TOWNSEND, F.R.S.)—For a system of quadrics inscribed in the same pair of cones, real or imaginary, and having consequently double contact with each other at the extremities of the chord common to the two planes of intersection of the cones, show that a variable line touching in every position the two cones determines (*a*) two systems of points inversely homographic to each other on every quadric of the system, (*b*) four systems of points, all homographic with each other on every two quadrics of the system, (*c*) pairs of variable chords cutting each other in constant anharmonic ratios in every pair of quadrics of the system, (*d*) triads of variable chords in involution with each other in every triad of quadrics of the system.

2632. (The EDITOR.)—Prove (1) that $1\,.\,2\,.\,3 \ldots n < 2^{\frac{1}{2}n(n-1)}$; and (2) that $\dfrac{27a^2b^2}{(a+b)^3} < 4a$ or $4b$, when a and b are both positive.

2664. (The late T. COTTERILL, M.A.)—1. Five points (no three in the same line, and no four in the same plane) determine, by the lines and planes through them on a plane, a system of ten points and ten lines, the points lying in threes on the lines, and the lines passing in threes through the points (CAYLEY). Show that the figure is its own polar reciprocal to a conic; and that, if a conic and triangle in its plane are given, the rest of the figure can be constructed.

2. A quadric through the five points cuts the plane in a conic containing triangles conjugate to the fixed conic. There is a quadric passing through the fixed conic, to which one of the five points and the plane are pole and polar, the remaining four points forming a self-conjugate tetrahedron.

2672. (R. TUCKER, M.A.)—From a point on an ellipse chords are drawn parallel to fixed straight lines; find the maximum triangle formed by the three chords of section.

2683. (R. TUCKER, M.A.)—To each point on the circumscribing circle of a triangle corresponds a foot-perpendicular line; this cuts the circle in two points; required the locus of the intersection of the feet-perpendicular lines corresponding to these points of section.

APPENDIX IV.

NOTES, SOLUTIONS, AND QUESTIONS.

BY R. W. D. CHRISTIE.

(A.) DIOPHANTINE ANALYSIS.

This branch of Algebra derives its name from its inventor, Diophantus of Alexandria, in Egypt, who flourished in or about the third century.

"No person has ever surpassed Diophantus in the solution of these problems, and few have equalled him."

1. *To find three square integers whose sum is a square integer.*

Let x^2, y^2, z^2 be the three required squares, and let $x^2+y^2 = a^2$.

Assume $a^2+z^2 = (na-z)^2$, then $a = (2nz)/(n^2-1)$.

Let $z = n^2-1$, then $a = 2n$.

Similarly, $y = m^2-1$, $x = 2m$, $a = m^2+1 = 2n$.

Therefore the squares are $(2m)^2$, $(m^2-1)^2$, $\left\{(m^2-1)(m^2+3)/4\right\}^2$, where m may be anything, but, if integers are desired, then m must be an odd number. Otherwise, let $(2ny)^2$, $\left\{y(n^2-1)\right\}^2$, and x^2 be the required squares. Assume $(2ny)^2+\left\{y(n^2-1)\right\}^2+x^2 = \left\{(n^2+1)y\right\}^2+x^2 = (ny+x)^2$, say. Then $x = \left\{\dfrac{(n^2+1)^2-n^2}{2n}\right\}y$. Let $y = 2n$; then $x = n^4+n^2+1$, and the three squares are $(2n)^4$, $\left\{2n(n^2-1)\right\}^2$, $\left\{n^4+n^2+1\right\}^2$, where n is anything.

It is clear the process may be extended to 4, 5, 6, &c. squares, but the following method is simpler.

2. *Find four integers in Arithmetical Progression, the sum of whose squares is a square integer.*

Let $x-1$, x, $x+1$, $x+2$ be the required numbers.

Assume $(x-1)^2+(x^2)+(x+1)^2+(x+2)^2 = 4x^2+4x+6 = (2x-y)^2$, say.

Then $x = \dfrac{y^2-6}{4(y+1)}$, where y is anything integral or fractional.

Let $y = 6$, then $x = \frac{15}{14}$, and the four squares are $(\frac{1}{14})^2$, $(\frac{15}{14})^2$, $(\frac{29}{14})^2$, $(\frac{43}{14})^2$, or, rejecting denominator, $1^2+15^2+29^2+43^2 = 54^2$.

It is plain the process may be extended to any (n^2) squares, *e.g.*,

$$2^2+5^2+8^2+11^2+14^2+17^2+20^2+23^2+26^2 = 48^2.$$

3. *To find three square numbers in Arithmetical Progression.*

Let a^2 be the first square, and $2ax + x^2$ the progression.

Then the three squares are a^2; $a^2 + 2ax + x^2$; $a^2 + 4ax + 2x^2$.

The first two are already squares. It remains to make

$$a^2 + 4ax + 2x^2 = \text{a square} = (nx - a)^2, \text{ say.}$$

Then $$x = \frac{2a(n+2)}{n^2-2}.$$

Let $a = n^2 - 2$; then $x = 2(n+2)$;

$$2ax + x^2 = \text{com. diff.} = (4n)(n+1)(n+2),$$

and the required squares are $(n^2-2)^2$; $(n^2+2n+2)^2$; $(n^2+4n+2)^2$, when n is anything.

4. If $N = a^2 + b^2$, prove that it also equals

$$\left\{\frac{2mnb + (n^2-m^2)a}{m^2+n^2}\right\}^2 + \left\{\frac{2mna + (m^2-n^2)b}{m^2+n^2}\right\}^2,$$

where a, b, m, and n may be anything integral or fractional.

Let $nx - a$, and $mx - b$ = sides of squares sought.

Then $(nx-a)^2 + (mx-b)^2 = N$. Therefore $(n^2+m^2)x = 2(mb+na)$,

$$\therefore\ nx - a = \left\{\frac{2mnb + (n^2-m^2)a}{m^2+n^2}\right\}, \text{ and } mx - b = \left\{\frac{2mna + (m^2-n^2)b}{m^2+n^2}\right\}.$$

Let a be any of the following expressions, viz.,

$$2n,\ 4(n+1),\ 4(2n-1),\ 3(2n+3),\ 12(n+1),\ 8(2n+3),\ 5(2n+5),$$
$$7(2n+9),\ 10(n+5),\ \{m(2n+m)\},\ \text{\&c. \&c.}$$

Similarly, b the corresponding expressions

$$(n^2-1),\ \{(2n+1)(2n+3)\},\ (4n^2-4n-3),\ 2n(n+3),\ (4n^2+8n-5),$$
$$(4n^2+12n-7),\ 2n(n+5),\ \{2(n+1)(n+8)\},\ n(n+10),\ \{2n(n+m)\},$$
$$\text{\&c.} \qquad \text{\&c.}$$

Then N becomes a square, and we shall have divided the sum of two squares into two other squares, N also being a square.

5. We know, by EULER's theorem, that a system of any number of binomial factors being multiplied together, their product is the sum of two squares, *e.g.*, $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = p^2+q^2$.

If, now, we make $ad = bc$ and $fg = eh$, q vanishes, and thus we can divide a square into four different pairs of squares, as *e.g.*,

$$1230^2 = 1200^2 + 270^2 = 738^2 + 984^2 = 1122^2 + 504^2 = 798^2 + 936^2.$$

6. *To divide a given square number which equals the difference of two square numbers, into the difference of two other squares.*

Generalising the expressions given above, we easily obtain

$$\{2r^2 + 2r(2n+1) + 4n\}^2 + \{(2n-1)(2r+2n+1)\}^2$$
$$= \{2r^2 + (2n+1)2r + 4n^2 + 1\}^2,$$

where n and r are anything.

In this equation make $n = 2$, $n = 1$ respectively, and we get

$$(2r^2+10r+8)^2 + (6r+15)^2 = \{2r^2+10r+17\}^2 \ \ldots\ldots\ldots\ldots \ (1),$$

and $$(2r^2+6r+4)^2 + (2r+3)^2 = \{2r^2+6r+5\}^2 \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

Now, equate the shortest sides $2s+3 = 6s+15$, say.

And let $2s+3$ = side of given square = 21, say, then $s = 9$, $r = 1$.

Then in (1) make $r = 1$, and in (2) $r = 9$.

Thus $20^2+21^2 = 29^2$, or $21^2 = 29^2-20^2$ .... (1).

And $220^2+21^2 = 221^2$, or $21^2 = 221^2-220^2$ .... (2).

Similarly for any other square whose side is given.

7. *Draw a straight line cutting two concentric circles so that the part intercepted by them is divided into three equal portions.*

Let R, r be the radii of the outer and inner circles respectively. As the outer intercepts are always equal, let x be the middle intercept. Then we must have $R^2-r^2 = 2x^2$.

The analysis gives us $x = \dfrac{2mr}{m^2-2}$.

Let $r = m^2-2$; then $x = 2m$, and $R = m^2+2$; *e.g.*, if $m = 4$ inches, then r, x, $R = 7, 4, 9$ inches. Therefore, measure off 4 inches from the outer circumference cutting the inner one, and produce the straight line joining the points. The three intercepts are equal.

8. (Question 2814.)—*To find three rational square integers in Arithmetical Progression having a common difference of* 13.

We have

$$\left\{x^2+(x+1)^2\right\}\left\{(x+1)^2+(x+2)^2\right\} \pm \left\{2(x+1)\right\}^2 = (2x^2+4x+3)^2$$

or

$$(2x^2+4x+1)^2 \text{ ....} (1).$$

Also $(m^2+n^2)^2 \pm (m+n)(m-n)(4mn) = \left\{m^2 \pm 2mn - n^2\right\}^2$ .... (2).

Now $13 = 2^2+3^2$; therefore let $x = 2$. Thus (1) gives us

$$13 \times 25 + 36 = 19^2 \quad \text{and} \quad 13 \times 25 - 36 = 17^2.$$

Let $p = 13$, $q^2 = 25$, $r^2 = 36$, and put pq^2 for m and r^2 for n in (2);

then $(p^2q^4+r^4)^2 \pm (pq^2+r^2)(pq^2-r^2)(4pq^2r^2)$ = two squares.

But $(p^2q^4-r^4)(4q^2r^2)$ = a square.

therefore $p^2q^4-r^4$ = a square, and consequently

$$\frac{p^2q^4+r^4}{(p^2q^4-r^4)\,4q^2r^2} = 30\,\frac{164568241}{375584400} = \text{square required,}$$

and the other two $= \dfrac{p^2q^4+r^4}{(p^2q^4-r^4)(4q^2r^2)} \pm 13.$

Hence it appears that the question will always admit of a solution when the given number plus or minus a square are both squares.

9. *To find a number which, being added to or subtracted from a square, the sum or remainder shall be a square.*

Use theorem 2 in (8), *supra*.

Let $m = 2$, $n = 1$; then $5^2 \pm 24 = 7^2$ or 1^2.

N.B.—The three squares are in Arithmetical Progression, thus $1^2 : 5^2 : 7^2$.

10. ***Construct a parallelogram whose sides and diagonals may be represented by integers.***

Let a, b be the two contiguous sides, and c, d the diagonals; then

$$(a+b)^2+(a-b)^2 = c^2+d^2.$$

but $\{p(p^2-3q^2)\}^2+\{q(3p^2-q^2)\}^2 = \{p(p^2+q^2)\}^2+\{q(p^2+q^2)\}^2.$

Assume p, q = any integer, and a, b, c, d are now easily found.

11. ***Two chords within or without a circle intersect at right angles. Construct it so that the four segments as well as the diameter may be represented by integers.***

Resolve any square into four other squares; *e.g.*,

$$D^2 = a^2+b^2+c^2+d^2 \text{ (v. infra).}$$

Then D = the diameter; a, b, c, d = the four segments of the intersecting chords.

12. ***Construct a regular decagon and a regular pentagon in a circle so that the sides of each, as well as the radius, may be represented by integers.***

Let P = side of pentagon, and D = side of decagon, R = radius. Then $P^2 = R^2+D^2$. Therefore assume any of the expressions given in A. 4 for R and D, making n = any integer.

13. ***Construct a series the terms of which may be taken to represent the three sides of a right-angled triangle, and find the sum.***

We have $\quad (2n-1)^2+(2n.n-1)^2 = (2n^2-2n+1)^2.$

Therefore assume the n^{th} term $= (2n-1)(2n.n-1)(2n^2-2n+1)$;

and the series becomes $1.0.1 + 3.4.5 + 5.12.13 + 7.24.25$, &c.,

and the sum $\quad = \frac{1}{2}n^2\{(4n^2-1)(n^2-1)\}.$

14. ***To find a series of biquadrates equal to a series of squares.***

We have $\quad \Sigma^4_{n=1(m.m+1)} / \Sigma^2_{1(m.m+1)} = (3n^2+3n-1)/5.$

If $n = 6$, we have

$$1^4+2^4+3^4+4^4+5^4+6^4 = 5^2(1^2+2^2+3^2+4^2+5^2+6^2).$$

From this value of n others may be obtained thus:—

Suppose that $x=f$, $y=g$ is a solution of the equation $x^2-Ny^2 = a$, and let $x = h$, $g = k$ be any solution of the equation $x^2-Ny^2 = 1$; then

$$x^2-Ny^2 = (f^2-Ng^2)(h^2-Nk^2) = (fh+Ngk)^2-N(fk\pm gh)^2.$$

By putting $\quad x = fh\pm Ngk, \quad y = fk\pm gh,$

and ascribing to h, k their values found by convergents, &c.

15. For $\Sigma^2 = 1^5+2^5+3^5 \dots n^5$, see an interesting solution in Vol. XLIX.

(B.) RESOLUTION OF SQUARES.

1. If any number N can be resolved into the sum of n squares, then $2(n-1)$ N can be resolved into the sum of $n(n-1)$ squares.

Let $N = a^2 + b^2$, then $4N = (2a)^2 + (2b)^2$.

Again, if $M = \alpha^2 + \beta^2 + \gamma^2$, then

$$4M = (\alpha+\beta)^2 + (\alpha-\beta)^2 + (\alpha+\gamma)^2 + (\alpha-\gamma)^2 + (\beta+\gamma)^2 + (\beta-\gamma)^2.$$

If $a^2 = \alpha^2 + \beta^2$, by any of the formulæ in (4), we easily get

$$a^2 + b^2 = \alpha^2 + \beta^2 + \gamma^2$$

$$= [\tfrac{1}{2}(\alpha+\beta)]^2 + [\tfrac{1}{2}(\alpha-\beta)]^2 + [\tfrac{1}{2}(\alpha+\gamma)]^2 + [\tfrac{1}{2}(\alpha-\gamma)]^2 + [\tfrac{1}{2}(\beta+\gamma)]^2 + [\tfrac{1}{2}(\beta-\gamma)]^2.$$

Ex. gr.: $10^2 + 2^2 = 6^2 + 8^2 + 2^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 7^2$.

Again, let $N = a^2 + b^2 + c^2$

$$= [\tfrac{1}{2}(a+b+c)]^2 + [\tfrac{1}{2}(a+b-c)]^2 + [\tfrac{1}{2}(a-b+c)]^2 + [\tfrac{1}{2}(b+c-a)]^2$$

$$= [\tfrac{1}{2}(a+b)]^2 + [\tfrac{1}{2}(a-b)]^2 + [\tfrac{1}{2}(a+c)]^2 + [\tfrac{1}{2}(a-c)]^2 + [\tfrac{1}{2}(b+c)]^2 + [\tfrac{1}{2}(b-c)]^2.$$

Ex. gr.: $N = 12^2 + 6^2 + 2^2 = 10^2 + 8^2 + 4^2 + 2^2 = 9^2 + 3^2 + 7^2 + 5^2 + 4^2 + 2^2$.

2. If any number N can be resolved into the sum of n squares, $\tfrac{1}{2}(n-1 \,.\, n-2) \,.\, N$ can be resolved into the sum of $4n$ squares if $n > 3$.

Let $N = a^2 + b^2 + c^2 + d^2$.

$$\begin{aligned} \text{Then } 3N = {} & [\tfrac{1}{2}(a+b+c)]^2 + [\tfrac{1}{2}(a+b-c)]^2 + [\tfrac{1}{2}(a-b+c)]^2 + [\tfrac{1}{2}(b+c-a)]^2 \\ & + [\tfrac{1}{2}(a+b+d)]^2 + [\tfrac{1}{2}(a+b-d)]^2 + [\tfrac{1}{2}(a-b+d)]^2 + [\tfrac{1}{2}(b+d-a)]^2 \\ & + [\tfrac{1}{2}(a+c+d)]^2 + [\tfrac{1}{2}(a+c-d)]^2 + [\tfrac{1}{2}(a-c+d)]^2 + [\tfrac{1}{2}(c+d-a)]^2 \\ & + [\tfrac{1}{2}(b+c+d)]^2 + [\tfrac{1}{2}(b+c-d)]^2 + [\tfrac{1}{2}(b-c+d)]^2 + [\tfrac{1}{2}(c+d-b)]^2. \end{aligned}$$

Similarly, 6N = sum of 20 squares, 10N = 24 (or 30 squares), &c. &c.

And, since the sum of the first n natural members is a perfect square, if n is $= k^2$ or $k'^2 - 1$, where k is the numerator of an odd, and k' the numerator of an even convergent of $\sqrt{2}$, if either (1) $\tfrac{1}{2}(n-1 \,.\, n-2) = N$; or (2) N and $\tfrac{1}{2}(n-1 \,.\, n-2)$ (a sequence from unity) are *both* squares, we can thus resolve a square into 16, 20, 24 ... $4(n+3)$ squares.

3. We know that $(a+b)^2 + (b+c)^2 + (c+a)^2 = (a+b+c)^2 + a^2 + b^2 + c^2 = M$.

If $a = \tfrac{1}{4}(3b-c)$, we have also $[\tfrac{1}{4}(7b-c)]^2 + [\tfrac{5}{4}(b+c)]^2 = M$.

If also we make $5(b+c) : 7b-c :: m : n$, where $m^2 + n^2 = p^2$, we get

$$\begin{aligned} M^2 = [\tfrac{1}{4}(7b-c)]^2 + [\tfrac{5}{4}(b+c)]^2 &= (a+b)^2 + (b+c)^2 + (c+a)^2 \\ &= (a+b+c)^2 + a^2 + b^2 + c^2. \end{aligned}$$

Or again, $M^2 = [\tfrac{1}{5}(7a+17b)]^2 + [\tfrac{12}{5}(a+b)]^2 = 3$ squares $= 4$ squares, &c.

Ex. gr.: $130^2 = 50^2 + 120^2 = 50^2 + 96^2 + 72^2 = 109^2 + 13^2 + 37^2 + 59^2$

= five squares, &c.

$= 78^2 + 104^2 = 40^2 + 78^2 + 96^2 = 107^2 + 11^2 + 29^2 + 67^2$, &c.

Generally, to get
$$\begin{aligned} N^2 = x^2 + y^2 &= (a+b)^2 + (a+c)^2 + (b+c)^2 \\ &= (a+b+c)^2 + a^2 + b^2 + c^2, \end{aligned}$$

let $(a+b) : (b+c) :: m : n$, where $m^2 + n^2 = p^2$.

Then $a = \dfrac{m(b+c) - nb}{n}$, and $(a+b)^2 + (a+c)^2 + (b+c)^2$

$$= \frac{(m^2+n^2)(b+c)^2}{n^2} + \left\{\frac{(m-n)\,b + (m+n)\,c}{m}\right\}^2$$

$$= \left\{\frac{p(b+c)}{n}\right\}^2 + \left\{\frac{(m-n)\,b + (m+n)\,c}{n}\right\}^2 = x^2 + y^2;$$

and to make $x^2+y^2=N^2$, assume

$$p(b+c):(m-n)\,b+(m+n)\,c::q:r,\ \text{where } q^2+r^2=s^2,$$

and we get $c=pr-(m-n)\,q,\ b=(m+n)\,q-pr,\ a=(m-n)\,q+pr.$

Thus finally, $N^2=(2ps)^2=(2pq)^2+(2pr)^2=(2mq)^2+(2pr)^2+(2nq)^2$
$=\{pr+(m+n)q\}^2+\{(m-n)q+pr\}^2+\{(m+n)q-pr\}^2+\{pr-(m-n)\,q\}^2.$

And since m, n; q, r; p, s are interchangeable, we can always secure four different resolutions of N^2.

Ex. gr.:

$$\begin{aligned}130^2 &= 120^2+50^2 = 96^2+50^2+72^2 = 109^2+37^2+13^2+59^2\\ &= 50^2+120^2 = 30^2+120^2+40^2 = 95^2+65^2+25^2+55^2\\ &= 104^2+78^2 = 40^2+78^2+96^2 = 107^2+11^2+29^2+67^2\\ &= 78^2+104^2 = 30^2+104^2+72 = 103^2+31^2+1^2+73^2.\end{aligned}$$

The following transformations of squares may here be noticed:—

(1) $$N^2=\left\{\frac{2rN}{r^2+1}\right\}^2+\left\{\frac{Nr^2-N}{r^2+1}\right\}^2,$$

where N and r may be integers > unity. Or, again

$$N^2=N^2\left\{\frac{m^2-r^2}{m^2+r^2}\right\}^2+N^2\left\{\frac{2mr}{m^2+r^2}\right\}^2,$$

where m and r may be assumed at pleasure $m>r$.

(2) $(a^2+b^2)(c^2+d^2)=(ac\pm bd)^2+(ad\mp bc)^2=(ac)^2+(bc)^2+(ad)^2+(bd)^2,$

$(a^2+b^2)(c^2+d^2)(e^2+f^2)=(A^2+B^2)(e^2+f^2)=(Ae\pm Bf)^2+(Be\mp Af)^2,$

$$\begin{aligned}(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)&=(A^2+B^2)(C^2+D^2)\\ &=(AC\pm BD)^2+(BC\mp AD)^2=p^2+q^2,\end{aligned}$$

where
$$\begin{aligned}p&=AC+BD=(ac+bd)\,(eg+fh)+(bc-ad)\,(fg-eh),\\ q&=BC-AD=(bc-ad)\,(eg+fh)-(ac+bd)\,(fg-eh),\end{aligned}$$

is one of eight solutions.

(3) $$a^2+b^2+c^2=[\tfrac{1}{2}(a+b+c)]^2+[\tfrac{1}{2}(a+b-c)]^2+[\tfrac{1}{2}(a-b+c)]^2+[\tfrac{1}{2}(b+c-a)]^2 \ \ldots\ldots\ (1),$$

and $$a^2+b^2+c^2+d^2=[\tfrac{1}{2}(a+b+c+d)]^2+[\tfrac{1}{2}(a-b-c+d)]^2+[\tfrac{1}{2}(a-b+c-d)]^2+[\tfrac{1}{2}(a+b-c-d)]^2\ldots\ (2);$$

also $$(a+b+c)^2+a^2+b^2+c^2=(a+b)^2+(b+c)^2+(c+a)^2.$$

Thus (1), (2), $k^2+l^2+m^2+n^2=x^2+y^2+z^2+w^2+d^2$;

(1), (3), $p^2+q^2+r^2=x^2+y^2+z^2+w^2+(a+b+c)^2$;

(4) $$\begin{aligned}(n^4+3n^2+1)^2&=\{2n\,(n^2+1)\}^2+\{n^4+n^2+1\}^2\\ &=(2n)^4+\{2n\,(n^2-1)\}^2+(n^4+n^2+1)^2;\end{aligned}$$

(5) $$(y-z)^2+(z-x)^2+(x-y)^2+(x+y+z)^2=3\,(x^2+y^2+z^2).$$

Thus

$$a^2+b^2+c^2=\{\tfrac{1}{2}(y-z)\}^2+\{\tfrac{1}{2}(z-x)\}^2+\{\tfrac{1}{2}(x-y)\}^2+\{\tfrac{1}{2}(x+y+z)\}^2,$$

where x, y, z are the sides, and a, b, c the medians of any triangle. In this way other identities may be made practically useful.

(6) $$(cy-bz)^2+(az-cx)^2+(bx-ay)^2+(ax+by+cz)^2$$
$$=(ax)^2+(ay)^2+(az)^2+(bx)^2+(by)^2+(bz)^2+(cx)^2+(cy)^2+(cz)^2$$
$$=(a^2+b^2+c^2)\,(x^2+y^2+z^2).$$

4. By Pollock's Theorems we can resolve any number N into one, two, three, or four (not more) squares as follows :—

Take any trigonal number (or sequence from unity), say $p=10$.

We have $2p+1=21=m^2+m+1=(2n\pm 1)^2$, and the squares are $(n\pm 1)^2, n^2, n^2, n^2$; and $21=3^2+2^2+2^2+2^2$.

Again, if p = sum of two trigonal numbers, say 15; we have

$$2p+1=15=2a^2+2a+2b^2+1=(a+1)^2, a^2, b^2, b^2,$$

whose roots are $(a+1), (-a), (+b), (-b)=1$.

Thus $15=(3)^2+(-2)^2+(+1)^2+(-1)^2$ (roots $=1$).

Lastly, if p = sum of three trigonal numbers, say 47;

we have $2p+1=47=2a^2+2a+2b^2+4n^2\pm 2n+1,$

and the roots are $\mp a+n\pm 1, a-n, b+n, b-n, (=1)$.

Thus $47=(-3)^2+(+2)^2+(+5)^2+(-3)^2\ (=1)$.

Therefore, since every number must either be a trigonal number or composed of two or three trigonal numbers (Legendre, *Théorie des Nombres*), every odd number may be resolved into integral square numbers (not exceeding four) whose algebraic sum will be 1, 3, 5 ...... $2n-1$, up to the maximum. Even numbers may also be resolved into square numbers (not exceeding four) the algebraic sum of whose roots may always equal 2.

It may be interesting to reproduce here Pollock's method of resolving odd numbers into squares.

Resolve 57 into four squares the sum of whose roots may equal 9.

Then $2m+1=9$; thus $m=4$, and $m^2+m+1=4^2+4+1=21$.

Now

$57-21=36$, and $36+1=37=(-1)^2+(+2)^2+(+4)^2+(-4)^2$ (roots $=1$).

Therefore $21+36=57=\overset{2\ \ +2\ \ +2\ \ +2}{\overline{1^2+4^2+6^2+(-2)^2}}$ (roots $=9$).

By this theorem we can instantly transform one square into two, three, or four squares, *ad libitum*;

e.g., $181=1^2+4^2+8^2+10^2.$

Then $181-4^2=165=1^2+8^2+10^2=4^2+6^2+7^2+8^2;$

thus $1^2+10^2=101=4^2+6^2+7^2=1^2+6^2+8^2.$

Then $4^2+7^2=1^2+8^2=65=2^2+3^2+4^2+6^2;$

thus $7^2=2^2+3^2+6^2$, &c., &c.

5. We know, if $N=a^2+b^2+c^2+d^2$, there are two resolutions of 4N into

uneven squares, viz.,

$$4N = (a+b+c+d)^2+(a+b-c-d)^2+(a-b-c+d)^2+(-a+b-c+d)^2$$
$$= A^2+B^2+C^2+D^2, \text{ say} \ldots\ldots (1),$$
$$= (A-2d)^2+(B+2d)^2+(C+2c)^2+(D+2c)^2 \ldots\ldots (2).$$

E.g., let $N = 9^2 = 2^2+4^2+5^2+6^2 = \&c.$, by POLLOCK; then

$4N = 18^2 = 4^2+8^2+10^2+12^2 = 17^2+5^2+1^2+3^2 = 5^2+7^2+9^2+13^2$, by GAUSS.

6. In order to resolve M^2 into N integral squares, we may make use of the following principle:—

$$(n+1)^2-(n)^2 = 2n+1 = \text{any odd number} > 1,$$

therefore take any N squares whose sum $= 2n+1$,

say $$2^2+3^2 = 13 = 2n+1;$$

thus $n = 6$, then $7^2 = 6^2+2^2+3^2$, or generally

$$(n^2+n+1)^2 = (n \,.\, n+1)^2+(n+1)^2+n^2.$$

Or again, we have $1^2+2^2+3^2+4^2+5^2+6^2 = 91 = 2n+1$ and $n = 45$;

thus $45^2 = 44^2+1^2+2^2+3^2+4^2+5^2+6^2$, or generally

$$(3n^2+15n+28)^2 = (3n^2+15n+27)^2+ \\ n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2+(n+5)^2,$$

where n is arbitrary, and so on.

7. The same object may be effected thus:—We know by easy analysis

that if $$x = a^2-b^2, \quad y = 2ab,$$

then $$u^2 = x^2+y^2;$$

also, if $$x = a^2+b^2-c^2, \quad y = 2ac, \quad z = 2bc,$$

then $$u^2 = x^2+y^2+z^2;$$

again if $$x = a^2+b^2+c^2-d^2, \quad y = 2ad, \quad z = 2bd, \quad w = 2cd,$$

then $$u^2 = w^2+x^2+y^2+z^2;$$

and generally, if $$A_1 = a_1^2+a_2^2+a_3^2 \ldots - a_n^2,$$

and $$A_2 = 2a_1 a_n, \quad A_3 = 2a_2 a_n \ldots\ldots A_n = 2a_{n-1}a_n,$$

then $$U^2 = A_1^2+A_2^2+A_3^2 \ldots\ldots + A_n^2.$$

E.g., let $$a_1 = 1, \quad a_2 = 2 \ldots\ldots a_2 = 5;$$

then $$11^2 = 1^2+2^2+4^2+6^2+8^2,$$

and so on for any number of squares.

8. Numerous resolutions may be obtained thus:—Take the first n^2 natural numbers and arrange so. Let $n = 4$; then

1, 15, 3, 13, 5, 11, 7, 9,
16, 2, 14, 4, 12, 6, 10, 8.

Note the odd numbers in the top row and the even numbers in the bottom row, also the sum of the numbers at the top and bottom $= 17 = n^2+1$.

Take any four odd numbers whose sum = 32 from the top row, and the remaining four even numbers from the bottom row; *e.g.*,

$$5,\ 7,\ 9,\ 11,\quad 2,\ 4,\ 14,\ 16,$$
$$3,\ 7,\ 9,\ 13,\quad 2,\ 6,\ 12,\ 16.$$

The sum of the odd numbers $=32$, and the sum of the even numbers $=36$, and the total sum $=68=n\,.\,n^2+1$.

Then we have at once:—

$$\begin{aligned}
&(x+5)^2+(x+7)^2+(x+9)^2+(x+11)^2+(x+2)^2\\
&\qquad\qquad\qquad+(x+4)^2+(x+14)^2+(x+16)^2\\
=\ &(x+3)^2+(x+7)^2+(x+9)^2+(x+13)^2+(x+2)^2\\
&\qquad\qquad\qquad+(x+6)^2+(x+12)^2+(x+16)^2,\\
=\ &(x+1)^2+(x+15)^2+(x+14)^2+(x+4)^2+(x+12)^2\\
&\qquad\qquad\qquad+(x+6)^2+(x+7)^2+(x+9)^2,\\
=\ &(x+8)^2+(x+10)^2+(x+11)^2+(x+5)^2+(x+13)^2\\
&\qquad\qquad\qquad+(x+3)^2+(x+2)^2+(x+16)^2,\\
=\ &(x+1)^2+(x+12)^2+(x+8)^2+(x+13)^2+(x+15)^2\\
&\qquad\qquad\qquad+(x+6)^2+(x+10)^2+(x+3)^2,\\
=\ &(x+1)^2+(x+6)^2+(x+11)^2+(x+16)^2+(x+4)^2\\
&\qquad\qquad\qquad+(x+7)^2+(x+10)^2+(x+13)^2,\\
=\ &(x+15)^2+(x+14)^2+(x+12)^2+(x+9)^2+(x+3)^2\\
&\qquad\qquad\qquad+(x+2)^2+(x+5)^2+(x+8)^2,\\
=\ &(x+11)^2+(x+16)^2+(x+10)^2+(x+13)^2+(x+1)^2\\
&\qquad\qquad\qquad+(x+4)^2+(x+6)^2+(x+7)^2,\\
=\ &(x+11)^2+(x+16)^2+(x+4)^2+(x+7)^2+(x+14)^2\\
&\qquad\qquad\qquad+(x+2)^2+(x+9)^2+(x+5)^2,
\end{aligned}$$

&c., &c.

But as these squares are not all different, we get, after elimination of the identical squares,—

(1) $(x+14)^2+(x+4)^2+(x+7)^2+(x+9)^2$
$$=(x+8)^2+(x+13)^2+(x+10)^2+(x+3)^2=4x^2+68x+342,$$

(2) $(x+1)^2+(x+15)^2+(x+12)^2+(x+6)^2$
$$=(x+11)^2+(x+2)^2+(x+5)^2+(x+16)^2=4x^2+68x+406,$$

(3) $(x+15)^2+(x+14)^2+(x+12)^2+(x+9)^2$
$$=(x+11)^2+(x+16)^2+(x+10)^2+(x+13)^2=4x^2+100x+646,$$

(4) $(x+3)^2+(x+2)^2+(x+5)^2+(x+8)^2$
$$=(x+1)^2+(x+4)^2+(x+6)^2+(x+7)^2=4x^2+36x+102,$$

(5) $(x+12)^2+(x+8)^2+(x+15)^2+(x+3)^2$
$$=(x+11)^2+(x+16)^2+(x+4)^2+(x+7^2)=4x^2+76x+442,$$

(6) $(x+14)^2+(x+2)^2+(x+9)^2+(x+5)^2$
$$=(x+1)^2+(x+6)^2+(x+10)^2+(x+13)^2=4x^2+60x+306,$$

&c., &c.

It is now easy to make any of these = a square.

We have (1) $\quad 4x^2+68x+342 = (2x+17)^2+53 = y^2+53 = z^2$, say;
and, since $53 = 27+26$, we have at once $z = 27$, $y = 26$, and $x = \frac{9}{2}$.
From this value of x we easily obtain

$$54^2 = 17^2+23^2+27^2+37^2 = 15^2+25^2+29^2+35^2.$$

Similarly (2), $4x^2+68x+406$; if $x = \frac{41}{2}$, we establish

$$118^2 = 43^2+71^2+65^2+53^2 = 63^2+45^2+51^2+73^2.$$

(3) gives us $\quad 22^2 = 5^2+7^2+11^2+17^2 = 3^2+9^2+13^2+15^2,$

and (4) the same. In order, therefore, to find another value of x we have by EULER's method $x = y+\frac{1}{2}$. Thus

$$4x^2+36x+102 = 4y^2+40y+121 = (py-11)^2, \text{ say.}$$

Then $y = \frac{32}{3}$, and $x = \frac{67}{6}$.

Neglecting the denominator, we have

$$190^2 = 79^2+85^2+97^2+115^2 = 73^2+91^2+103^2+109^2.$$

In this way endless resolutions may be obtained.

$n = 6$, by a similar process, will give us a still larger variety of equations, of which one is

$$\begin{aligned}&(x+1)^2+(x+33)^2+(x+7)^2+(x+27)^2+(x+22)^2+(x+21)^2\\ &= (x+36)^2+(x+4)^2+(x+30)^2+(x+10)^2+(x+15)^2+(x+16)^2, \text{ \&c., \&c.}\end{aligned}$$

If $n = 10$, we may obtain any number of resolutions such as

$$\begin{aligned}&(x+100)^2+(x+98)^2+(x+92)^2+(x+86)^2+(x+4)^2+(x+5)^2+(x+6)^2\\ &\qquad +(x+7)^2+(x+8)^2+(x+99)^2\\ &= (x+1)^2+(x+3)^2+(x+9)^2+(x+15)^2+(x+97)^2+(x+96)^2+(x+95)^2\\ &\qquad +(x+94)^2+(x+93)^2+(x+2)^2,\end{aligned}$$

or, if we prefer it,

$$\begin{aligned}&(2)^2+(2n^2-n)^2+(2n+2)^2+(2n^2-3n)^2+(4n+2)^2+(2n^2-5n)^2+(6n+2)^2\\ &\qquad +(2n^2-7n)^2+(13n+2)^2+(2n^2-9n)^2\\ &= (2n^2)^2+(n+2)^2+(2n^2-2n)^2+(3n+2)^2+(2n^2-4n)^2+(5n+2)^2\\ &\qquad +(2n^2-6n)^2+(7n+2)^2+(2n^2-13n)^2+(9n+2)^2,\end{aligned}$$

the principle of which will readily suggest itself on a comparison of the upper and lower squares; and generally, for any unevenly even number of squares, say 18, we have

$$\begin{aligned}&(2)^2+(2n^2-n)^2+(2n+2)^2+(2n^2-3n)^2+(4n+2)^2+(2n^2-5n)^2+(6n+2)^2\\ &\quad +(2n^2-7n)^2+(8n+2)^2+(2n^2-9n)^2+(10n+2)^2+(2n^2-11n)^2+(12n+2)^2\\ &\quad +(2n^2-13n)^2+(14n+2)^2+(2n^2-15n)^2+(25n+2)^2+(2n^2-17n)^2\\ &=(2n^2)^2+(n+2)^2+(2n^2-2n)^2+(3n+2)^2+(2n^2-4n)^2+(5n+2)^2+(2n^2-6n)^2\\ &\quad +(7n+2)^2+(2n^2-8n)^2+(9n+2)^2+(2n^2-10n)^2+(11n+2)^2+(2n^2-12n)^2\\ &\quad +(13n+2)^2+(2n^2-14n)^2+(15n+2)^2+(2n^2-25n)^2+(17n+2)^2.\end{aligned}$$

9. To make Σ_n^3/Σ_n^1 = a square, where

$$\Sigma_n^r = 1^r + (a+1)^r + (a+2)^r \ldots\ldots (a+n-1)^r,$$

and a the common difference, we have

$$\Sigma_n^3/\Sigma_n^1 = 1 + a(n-1) + a^2[\tfrac{1}{2}(n\,.\,n-1)] = M^2.$$

($a = 1$.) Then $n^2+n = 2M^2$; thus $8M^2+1$ = a square = N^2.

Thus $8M^2-N^2 = -1$, an equation easily solved by the even convergents of $\sqrt{2}$.

($a = 2$.) Then $2n^2-1 = M^2$, solved by the odd convergents of $\sqrt{2}$; *e.g.*, $n = 5$, $M = 7$; thus $\dfrac{1^3+3^3+5^3+7^3+9^3}{1+3+5+7+9} = 7^2$.

Consequently, $1^3+3^3+5^3 \ldots (2n-1)^3 = (nM)^2$, where n is the denominator and M the numerator of any odd convergent of $\sqrt{2}$.

($a = 3$.) We have $9n^2-3n-4 = 2M^2$, or $(3n-\tfrac{1}{2})^2 = \tfrac{1}{4}(8M^2+17)$, $M = 1$ obvious.

Therefore, let $M = N+1$; thus $8M^2+17 = 8N^2+16N+25 = (pN-5)^2$, say.

Thus $N = \dfrac{2(5p+8)}{p^2-8}$; $p = 4$, &c.; $N = 7$, &c.; $M = 8$, &c.; $n = 4$, &c.;

e.g., $$\frac{1^3+4^3+7^3+10^3}{1+4+7+10} = 8^2,$$

and so on for higher values of a.

When any one value of n is obtained, others are readily got by substitution.

In a similar way we can convert Σ_n^3/Σ_n^1 into a square, where

$$\Sigma_r^n = a^r + (a+1)^r + (a+2)^r \ldots\ldots (a+n-1)^r,$$

a being the first term; *e.g.*, $a = 3$ gives us, if $n^2-n+2a(n-1)+2a^2 = 2M^2$,

$$n^2+5n+12 = 2M^2 \text{ or } (n+\tfrac{5}{2})^2 = \tfrac{1}{4}(8M^2-23),\ M = 2 \text{ obvious.}$$

Therefore, let $M = N+2$; then $8M^2-23 = 8N^2+32N+9 = (pN-3)^2$, say.

Then $$N = \frac{2(3p+16)}{p^2-8},\ p = 4,\ N = 7,\ M = 9,\ n = 10.$$

Thus $$\frac{3^3+4^3+5^3+6^3+7^3+8^3+9^3+10^3+11^3+12^3}{3+4+5+6+7+8+9+10+11+12} = 9^2,$$

and so on for any value of a.

10. *To resolve a square into two, three, four, or more squares.*

Applying the results given above, we easily establish

$$(2n^4+4n^3+4n^2+2n+1)^2 = (2n^2+2n+1)^2 + \{(2n\,.\,n+1)(n^2+n+1)\}^2$$
$$= (2n+1)^2 + (2n\,.\,n+1)^2 + \{(2n\,.\,n+1)(n^2+n+1)\}^2$$
= four squares, by Pollock, = &c.

Ex. gr., ($n = 1$) $13^2 = 5^2+12^2 = 3^2+4^2+12^2 = 4^2+6^2+6^2+9^2 =$ &c.

11. Again, to resolve three squares into three others, the sums of whose roots are equal, we may make use of the following identity,

$$(b+cd)^2+(bc)^2+d^2 \equiv (bc+d)^2+b^2+(cd)^2, \text{ where } b,\ c,\ d \text{ are arbitrary,}$$

the sum of the roots being $(b+d)(c+1)$; and, if $b=2n$, $c=n+1$, $d=2n+1$, we easily get

$$(2n^2+5n+1)^2+(2n.n+1)^2+(2n+1)^2 = (2n^2+5n+1)^2+(2n^2+2n+1)^2$$
$$= (2n^2+4n+1)^2+(2n)^2+\{(n+1)(2n+1)\}^2.$$

Ex. gr., $8^2+4^2+3^2 = 8^2+5^2 = 7^2+2^2+6^2$,

$19^2+12^2+5^2 = 19^2+13^2 = 17^2+4^2+15^2$, &c.

Similarly $\{m(n+m)\}^2+\{n(n+m)\}^2+(mn)^2 = \{m^2+mn+n^2\}^2$.

12. *To resolve an even square into eight or sixteen squares.*

Primes are of two classes, viz., $4n+1$ and $4n-1$ form.

Now it has been demonstrated by De la Grange (*Memoirs of Berlin*, 1768), and others, that primes of the former class may be resolved into the sum of two squares; and, since it may be shown that every even square > 5 equals the sum of four $4n+1$ primes, it follows that every even square > 5 equals the sum of sixteen squares, since Pollock has shown that every odd number may be resolved into four squares; thus

$6^2 = 1+5+13+17$ = sixteen squares (by Pollock)
= eight squares (by Lagrange),

$8^2 = 5+13+17+29 =$ &c.,

$10^2 = 13+17+29+41 =$ &c.,
&c. &c.

$18^2 = 37+41+97+149 = 41+61+73+149 = 41+53+73+157$
$= 41+73+97+113 = 41+73+101+109 =$ &c., &c.,
= sixteen squares in six ways at least.

(C.) RESOLUTION OF CUBES.

Euler has demonstrated that it is impossible to find any two cubes whose sum or difference is a cube; also that the formula $x^3 \pm y^3 = 2z^3$ is impossible.

There are two methods of resolving a cube into three cubes.

(*a*) *If two of the cubes are given, and a third be required to make the three cubes equal a cube*, we may use the following formula:—

$$\{a(b^3-a^3)\}^3+\{b(b^3-a^3)\}^3+\{a(2b^3+a^3)\}^3 = \{b(b^3+2a^3)\}^3.$$

Thus any cube may be resolved into three cubes; thus,

$$x^3 = \left\{\frac{a(b^3-a^3)}{b(b^3+2a^3)}\right\}^3 x^3 + \left\{\frac{b(b^3-a^3)}{b(b^3+2a^3)}\right\}^3 x^3 + \left\{\frac{a(2b^3+a^3)}{b(b^3+2a^3)}\right\}^3 x^3,$$

and a similar principle applies throughout.

(*b*) Again, *if it be required in general to find three cubes whose sum may equal a cube*, EULER gives us

$$\begin{aligned} x &= p+q = (ft+3gu)+(gt-fu), \\ y &= p-q = (ft+3gu)-(gt-fu), \\ v &= r+s = (kt-hu)+(ht+3ku), \\ z &= r-s = (kt-hu)-(ht+3ku), \end{aligned}$$

where $u = f(f^2+3g^2)-h(h^2+3k^2)$, $t = 3k(h^2+3k^2)-3g(f^2+3g^2)$,
and f, g, h, k are arbitrary.

$$\begin{aligned} \therefore\ x^3+y^3 = v^3-z^3 &= \{(ft+3gu)+(gt-fu)\}^3+\{(ft+3gu)-(gt-fu)\}^3 \\ &= \{(kt-hu)+(ht+3ku)\}^3-\{(kt-hu)-(ht+3ku)\}^3. \end{aligned}$$

(*c*) *To obtain four cubes equal to a cube*, we may use the following equation:— $(s-a)^3+(s-b)^3+(s-c)^3+3abc = s^3$.

Assume $3abc = (3mn)^3$, thus $abc = 9m^3n^3$. Now $9m^3n^3 = abc$ may be assumed in any combination; *e.g.*, let $a = 9m$, $b = m^2n$, $c = n^2$. Thus, *e.g.*, if $m = 2$, $n = 3$, we easily get $1^3+5^3+7^3+12^3 = 13^3$.
Other combinations are $a = 9m^3$, $b = n$, $c = n^2$;
$a = 9m^2$, $b = mn$, $c = n^2$;
$a = 9m$, $b = m^3$, $c = n^3$, &c., &c.

To secure the desired result make $m<n$.

(*d*) If $m>n$, we may secure the result that *the sum of two cubes equals the sum of three cubes* thus: $m = 3$, $n = 2$ gives us

$$13^3+41^3+36^3 = 49^3+5^3.$$

(*e*) We may secure *four cubes equal to two cubes* by the use of the following identity:—

$$\begin{aligned} (x+y+z)^3+(x+y-z)^3+(x-y+z)^3+(x-y-z)^3 &= 4x(x^2+3y^2+3z^2) \\ = 4x^3+12xy^2+12xz^2 = x^3+3x(x^2+4y^2+4z^2) &= x^3+3x(\mathrm{M}^2), \end{aligned}$$

if $x = a^2+b^2-c^2$, $y = ac$, and $z = bc$, making $\mathrm{M} = a^2+b^2+c^2$.

If, also, we assume $3x = \mathrm{M}$, *i.e.* if $a^2+b^2 = 2c^2$, by making

$$a = p^2+2qp-q^2, \quad b = p^2-2qp-q^2, \quad c = p^2+q^2,$$

we establish $(x+y+z)^3+(x+y-z)^3+(x-y+z)^3+(x-y-z)^3 = x^3+(3x)^3$;

(*f*) Also, if x, y, and z are so chosen that any two are > the third, we have *the sum of three cubes resolved into three other cubes*, *e.g.*,

$$p = 2 \text{ or } 3,\ q = 1, \text{ gives us } 11^3+13^3 = 1^3+3^3+5^3+15^3,$$
$$p = 4,\ q = 1, \text{ gives us } 799^3+561^3+17^3 = 289^3+867^3+221^3.$$

(*g*) *To find n cubes whose sum may equal a square.*

Let $\Sigma_n^r = 1^r+2^r+3^r \dots + n^r$.

It is easy to prove that $\Sigma^3+(2\Sigma')^3 = (3\Sigma^2)^2$,

or $$(1^3+2^3+3^3 \dots +n^3)+(n^2+n)^3 = \left(\frac{n \, . \, n+1 \, . \, 2n+1}{1.2}\right)^2;$$

e.g., if $n = 5$, we have

$$1^3+2^3+3^3+4^3+5^3+30^3 = 165^2 = \{3(1^2+2^2+3^2+4^2+5^2)\}^2.$$

[$\Sigma_n^3 = (\Sigma_n')^2$ is well known.]

(*h*) *To find two cubes whose sum may equal the difference of two squares.*

Since $\Sigma^3 = (\Sigma')^2$, we get from (*g*)

$$(2\Sigma')^3 = (3\Sigma_2)^2 - (\Sigma')^2,$$

or
$$(n^2+n)^3 = \left(\frac{n\,.\,n+1\,.\,2n+1}{1\,.\,2}\right)^2 - \left(\frac{n^2+n}{2}\right)^2;$$

and, since we know that $n^3 = \left(\frac{n^2+n}{2}\right)^2 - \left(\frac{n^2-n}{2}\right)^2$,

we have at once $n^3 + (n^2+n)^3 = \left(\frac{n\,.\,n+1\,.\,2n+1}{1\,.\,2}\right)^2 - \left(\frac{n^2-n}{2}\right)^2.$

(*j*) *To find a number of odd cubes whose sum may equal a number of even cubes.*

We have $1^3-2^3+3^3-4^3 \ldots = -(4n^3+3n^2)$;

therefore $1^3+3^3+5^3 \ldots + (2n-1)^3 + 4N^3 + 3N^2 = 2^3+4^3+6^3 \ldots + (2N)^3$,

or $1^1+3^3+5^3 \ldots + N^3 + 3N^2(N+1) = 2^3+4^3+6^3 \ldots + (2N)^3.$

To make $3N^2(N+1) =$ a cube, let $N = 8$; thus

$$1^3+3^3+5^3+7^3+9^3+11^3+13^3+15^3 = 2^3+4^3+6^3+10^3+14^3+16^3.$$

To obtain other values of N, we may proceed by Euler's mode to substitute $N = M+8$ in $3N^2(N+1)$, and equate it to $(pm+12)^3$, and so on.

(*k*) *To resolve a cube into a number of squares.*

We know by the Analysis that x^2+y^2 can be made $= (p^2+q^2)^m$.

Let
$$x = p^3-3pq^2, \quad y = 3p^2q-q^3;$$

then
$$x^2+y^2 = (p^2+q^2)^3 = \{p(p^2+q^2)\}^2 + \{q(p^2+q^2)\}^2;$$

similarly

$$(p^2+q^2+r^2)^3 = \{p(p^2+q^2+r^2)\}^2 + \{q(p^2+q^2+r^2)\}^2 + \{r(p^2+q^2+r^2)\}^2,$$

and generally

$$\{p^2+q^2+r^2 \ldots z^2\}^3 = \{p(p^2+q^2+\ldots z^2)\}^2 + \{q(p^2+q^2+\ldots z^2)\} + \text{\&c.},$$

or
$$\left\{\frac{m\,.\,m+1\,.\,2m+1}{6}\right\}^3 = \left\{1^2\left(\frac{m\,.\,m+1\,.\,2m+1}{6}\right)\right\}^2 + \left\{2^2\left(\frac{m\,.\,m+1\,.\,2m+1}{6}\right)\right\}^2 + \text{\&c.}$$

(*l*) *Every cube may be resolved into an Arithmetical Progression.*

We have $\Sigma = a + (a+b) + (a+2b) \ldots a + (n-1)b = n\left\{a + \frac{n-1}{2}\,.\,b\right\}$.

Let $n = 2m+1$, then $\Sigma = (2m+1)(a+bm)$.

If $a+bm = (2m+1)^2$, then sum of n terms $= n^3$. Thus let

$$a = 2n-1, \quad b = 2(n-1), \quad n = 5;$$

e.g.,
$$9+17+25+33+41 = 5^3.$$

(*m*) *To resolve a number into three cubes.*

Take 6 for example.

Assume $$6 = (2-bx)^3+(cx-1)^3+(dx-1)^3,$$
and make $$b = (c+\ )/4;$$
then $$x = \{56(c^2+d^2)-16cd\} / \{21(c^3+d^3)-cd(c+d)\}.$$

If $c = 7$, $d = 5$, $b = 3$, then $6 = (\frac{6}{7})^3+(\frac{5}{3})^3+(\frac{19}{21})^3$.

Again, take $$8 = 2^3 = (2-bx)^3+(cx+1)^3+(dx-1)^3,$$
and make $$b = (c+d)/4;$$
then $$x = \{3(d^2-c^2)-6b^2\} / \{c^3+d^3-b^3\}.$$

If $d = 7$, $c = 5$, then $b = 3$ and $x = \frac{2}{49}$, and we have
$$35^3+98^3 = 92^3+59^3,$$
or the sum of two cubes resolved into two other cubes.

If $d = 5$, $c = 7$, $b = 3$, we get $x = -\frac{2}{7}$ and $20^3 = 7^3+14^3+17^3$, or a cube equal to three cubes.

Similarly, for any number if we take care to make the given number in the original equation vanish and to equate the coefficient of x to zero.

(*n*) *To find a cube equal to the difference of two squares.*

Assume $$x^2-y^2 = a^2 = (x-ny)^2, \text{ say};$$
then $$2nx = y(y+n^2).$$
If $y = 2n$, then $x = n^2+2n$ and $a = 2n-n^2$; thus
$$(2n)^3 = (n^2+2n)^2-(n^2-2n)^2,$$
where n is arbitrary.

(*o*) *To resolve the sum of four cubes or biquadrates into four other cubes or biquadrates.*

We have $(a+b+c)^n+a^n+b^n+c^n = (a+b)^n+(b+c)^n+(c+a)^n+X^n$,

$(n=3)$. $(a+b+c)^3+a^3+b^3+c^3 = (a+b)^3+(b+c)^3+(c+a)^3+6abc$.

To make $6abc = X^3$, proceed as in (C. *e*), and assume $6abc = (6mn)^3$ in any order, as, *e.g.*, $a = 36m$, $b^2 = m^2n$, $c = n^2$.

Thus $m = 2$, $n = 3$ give us $3^3+4^3+24^3+31^3 = 7^3+12^3+27^3+28^3$.

$(n = 4)$. Again, for
$$(a+b+c)^4+a^4+b^4+c^4 = (a+b)^4+(b+c)^4+(c+a)^4+12abc(a+b+c).$$
To make $$12abc(a+b+c) = X^4 = (nb)^4 \text{ say};$$
let $c = 1$, $a = 3bn^2$, thus $36(3n^2b+b+1) = (nb)^2$.

Now assume $$3n^2b+b+1 = (2nb-1)^2 \text{ say};$$
thus $b = \dfrac{n+1 . 3n+1}{4n^2}$, where n is arbitrary; *e.g.*, $n=1$ gives us
$$1^4+2^4+9^4 = 3^4+7^4+8^4.$$

(D.) SOLUTIONS OF OLD QUESTIONS.

1010. (The EDITOR.)—Find integral values of x, y, z which will make x^2-2y^2, y^2-3z^2, x^2-5z^2 all squares.

Solution.

An obvious solution is $x = 3z$, $y=2z$; but this makes two of the squares identical. Therefore, let $y = 2z$, then $x^2-8z^2 = a^2$, $x^2-5z^2 = b^2$; thus
$$a^2+3z^2 = b^2.$$

Let $a = p^2-3q^2$, $z = 2pq$; then $b = p^2+3q^2$.

To find x, we have $x^2 = a^2+8z^2 = b^2+5z^2 = p^4+26p^2q^2+9q$

$$= (p^2-5q)^2, \text{ say therefore, } 3p = 2q.$$

Thus $\quad x = p^2-5q^2,\ \ y = 4pq,\ \ z = 2pq,$ where $3p = 2q$;

e.g., $\quad p = 2,\ \ q = 3,\ \ x = 41,\ \ y = 24,\ \ z = 12;$

$$a^2 = 23^2,\quad b^2 = 31^2,\quad c^2 = 12.$$

1014. (The EDITOR.)—Find the least integral value of x which will make the expression $927x^2-1236x+413$ = a square.

Solution.

We have $103\,(3x-2)^2+1 = y^2$, *i.e.*, $y^2-103z^2 = 1$.

The twelfth convergent of $\sqrt{103}$ gives us $y = 227528$, $z = 22419$, and the 24th $y = 103537981567$, $z = 10201900464$; but neither of these values of z will give us an integral value of x; so that either we must go farther or else the method of convergents does not ensure us integral values.

If $x = \frac{2}{3}$, we have $\quad 927x^2-1236x+413 = 1.$

Suppose the quantity $a+bx+cx^2 = g^2$, where $x = f$, so that we have
$$a+bf+cf^2 = g^2,$$
then we get
$$x = \tfrac{2}{3}.\left(\frac{m^2-3m-927}{m^2-927}\right),$$
where m is arbitrary, an equation giving all other possible values of x; whence it is clear x can never be an integer.

1042. (The EDITOR.)—Find values of x which will make each of the expressions $3x^3+1$, x^3+1, $2x^4-3x^2+2$ a square number.

Solution.

EULER has shown that x^3+1 can never become a square except in three cases, viz., $x = 0$, -1, and 2; so that, although we may readily obtain values for x in the other two equations, it would appear that no value of x will simultaneously satisfy the three equations.

2814. (The late MATTHEW COLLINS, B.A.)—Can the common difference of three rational square integers in Arithmetical Progression be ever equal to 17?

Solution.

Square numbers are of two forms, 4N and 4N + 1.

Let the three required squares be x^2, y^2, z^2.

Suppose (1) $x^2 = 4N$;

then $y^2 = x^2+17 = 4N+17 = 4M+1$, which is possible,

$z^2 = x^2-17 = 4N-17 = 4M-1$, ,, impossible.

Again (2), let $x^2 = 4N+1$;

then $y^2 = 4N+18 = 4M+2$, which is impossible.

Thus, by either supposition it is impossible for x^2, y^2, z^2 to be squares simultaneously.

8930. (R. W. D. CHRISTIE.)—Prove that, whether (n) be odd or even,

$$\sin n\theta = \sin\theta\left\{(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5}\right.$$
$$\left.-\frac{(n-4)(n-5)(n-6)}{3!}(2\cos\theta)^{n-7}+\dots\right\}.$$

*Solution.**

Cette formule est certaine pour les cas $n = 2$, $n = 3$, parce que l'on a d'abord

$$\text{sen } 2\theta = 2\text{ sen }\theta\cos\theta = \text{sen }\theta\left\{(2\cos\theta)^{2-1}\right\},$$

$$\text{sen } 3\theta = \text{sen } 2\theta\cos\theta+\text{sen }\theta\cos 2\theta = 2\text{ sen }\theta\cos^2\theta+\text{sen }\theta(\cos^2\theta-\text{sen}^2\theta)$$
$$= \text{sen }\theta(4\cos^2\theta-1) = \text{sen }\theta\left\{(2\cos\theta)^2-(3-2)(2\cos\theta)^0\right\}$$
$$= \text{sen }\theta\left\{(2\cos\theta)^{3-1}-(3-2)(2\cos\theta)^{3-3}\right\}.$$

Cela posé, nous allons prouver que la formule qui est vérifiée pour le cas $(n-1)$, le sera pour le cas (n).

Par supposition nous avons

$$\text{sen }(n-1)\theta = \text{sen }\theta\left\{(2\cos\theta)^{n-2}-(n-3)(2\cos\theta)^{n-4}\right.$$
$$\left.+\frac{(n-4)(n-5)}{1.2}(2\cos\theta)^{n-6}\dots\right\}\dots\dots\dots (1).$$

Mais $\text{sen } n\theta = \text{sen }\theta.\cos(n-1)\theta+\cos\theta\text{ sen }(n-1)\theta$;

mais (voyez CARR, *Synopsis of Pure Mathematics*, page 177) l'on sait que

$$2\cos(n-1)\theta = (2\cos\theta)^{n-1}-(n-1)(2\cos\theta)^{n-3}+\frac{(n-1)(n-4)}{1.2}(2\cos\theta)^{n-5}\dots,$$

d'où $\cos(n-1)\theta$

$$= 2^{n-2}(\cos\theta)^{n-1}-(n-1)2^{n-4}(\cos\theta)^{n-3}+\frac{(n-1)(n-4)}{1.2}(\cos\theta)^{n-5}.2^{n-6}\dots,$$

et substituant la valeur de $\text{sen }(n-1)\theta$ en (1) nous avons très facilement la formule demandée.

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* This solution is due to Professors BEYENS and CATALAN.

9444. (R. W. D. CHRISTIE.)—Solve (1) in integers $x^4+x^2y^2+y^4=ab$; and (2) note the result when $a=b$.

Solution.

We have $(x^2+xy+y^2)(x^2-xy+y^2) = ab.$

Assume $x^2 \pm xy+y^2 = a$ or $b = x^2 \pm xy/n$;

then $x = \pm ny/n-1$. Let $y=n-1$; then $x=\pm n$; also $a=n^2-n+1$.

n may now be assumed any integer at leisure.

If $x^4+x^2y^2+y^4 = a^2 = (x^2-n^2y^2)^2$, say,

then $x^2(2n^2+1) = (n^4-1)y^2.$

If $y^2 = 2n^2+1.$

then $x^2 = n^4-1.$

9521. (R. W. D. CHRISTIE.)—Prove that $(p^4 \sim \pi^4)/5$ is an integer where p is *any* perfect number and π *any* prime number except 5.

Solution.

Perfect numbers end in the digits 6 or 8; therefore p^4 ends in 6.

Prime numbers end in 1, 3, 7, or 9 (except 2 and 5); therefore π^4 ends in unity, also 2^4 ends in 6.

Therefore $(p^4-\pi^4)/5$ is an integer, except $\pi=5$.

9608. (SEPTIMUS TEBAY, B.A.)—Find the least heptagonal number which when increased by a given square shall be a square number.

Solution.

The general form of heptagonal numbers is $\frac{1}{2}(5x^2-3x)$. Let a^2 be the given square, and h the number sought.

Assume $a^2+h = (a-n)^2$;

then $h = n^2-2an.$

But $40h+9$ is always a square; therefore assume

$$40(n^2-2an)+9 = (6n+3)^2.$$

Thus $n=20a+9$, and a may be assumed at pleasure. Let $a=1$, then $n=29$, therefore $h=29^2-58=783$. Thus $783+1=28^2$.

9629. (Professor GERONDAL.)—Partager 90° en deux parties x, y telles que la tangente de l'une soit le quadruple de la tangente de l'autre, et prouver que $\tan\frac{1}{2}x = 2\sin 18°$.

Solution.

We have $x+y = 90°$ .... (1),

and $\tan x = 4\tan y$ ....(2).

Then $\tan x \tan y = 1$ ....(1);

therefore $4\tan^2 y = 1$ ....(1), (2),

$\tan y = \pm\frac{1}{2}$,

$\tan x = \pm 2 = (2\tan\frac{1}{2}x)/(1-\tan^2\frac{1}{2}x)$;

therefore $\tan\frac{1}{2}x = \frac{1}{2}(\sqrt{5}-1) = 2\sin 18°$.

9643. (R. W. D. Christie.) — If $\Sigma_n^r \equiv 1^r + 2^r + 3^r \dots n^r$, prove that Σ_n^r is exactly divisible by Σ_n^1 when r is odd.

Solution.

Obtain Σ_n^r by the formula $(n+1)\,\Sigma_n = (du/dx)\,\Sigma_{n+1}$ and separate *odd* values of r from the evens, and we shall find that Σ_n^1 is a constant factor of the *odd* and $(2n+1)/2(r+1)$ of the even. Thus, let $s = \Sigma_n^1$; then

$r = 1$ gives us s,

$r = 3$ " s^2,

$r = 5$ " $\frac{1}{3}s^2(4s-1)$,

$r = 7$ " $\frac{1}{3}s^2\{6s^2-(4s-1)\}$,

$r = 9$ " $\frac{1}{5}s^2\{16s^3-20s^2+3(4s-1)\}$,

$r = 11$ " $\frac{1}{3}s^2\{16s^4-32s^3+34s^2-5(4s-1)\}$,

$r = 13$ " $\dfrac{1}{3.5.7}s^2\{960s^5-2800s^4+4592s^3-4720s^2+691(4s-1)\}$.

$r = 2$ " $\frac{1}{3}ks$,

$r = 4$ " $\frac{1}{5}ks\,[\frac{1}{3}(6s-1)]$,

$r = 6$ " $\frac{1}{7}ks\,[\frac{1}{3}(12s^2-6s+1)]$.

$r = 8$ " $\frac{1}{9}ks\,[\frac{1}{5}(40s^3-40s^2+18s-3)]$.

$r = 10$ " $\frac{1}{11}ks\,[\frac{1}{3}\{48s^4-80s^3+68s^2-5(6s-1)\}]$,

$r = 12$ " $\frac{1}{13}ks\left[\dfrac{1}{3.5.7}\{3360s^5-8400s^4+11480s^3-9440s^2+691(6s-1)\}\right]$,

$r = 14$ " $\frac{1}{15}ks\,[\frac{1}{3}\{192s^6-672s^5+1344s^4-1760s^3+1436s^2-105(6s-1)\}]$,

&c., &c.

Thus the theorem appears to be true for odd values of r only; that it is not true for even values of r may easily be tested by making $r = 4$, $n = 2$, 3, or 4, for example.

9668. (Professor Vuibert.)—Si l'on désigne d'une manière générale par S_m la somme des puissances de degré m des n premiers nombres entiers, démontrer qu'on a $(3S_5 + 2S_1^4)$, $5S_4 = S_3, S_2$.

9643. (R. W. D. Christie.)—If $\Sigma_n^r = 1^r+2^r+3^r \dots n^r$; prove that Σ_n^r is divisible by Σ_n^1.

9226. (J. White.)—Prove that

$$1^3+2^3+3^3 \dots M^3 \text{ is a factor of } (1^5+2^5+3^5 \dots M^5)\times 3.$$

8784. (R. W. D. Christie, M.A.)—Prove that, if

$$s = 1+2+3+\dots+n,\quad S^2 = 1^2+2^2+3^2+\dots+n^2,\quad S^3 = 1^3+2^3+3^3+\dots+n^3,$$
$$\Sigma = 1^4+2^4+3^4+\dots+n,\quad \sigma = 1^5+2^5+3^5+\dots+n^5,$$

then
$$(3\sigma+2s^3)/5\Sigma = S^3/S^2.$$

9383. (R. W. D. Christie, M.A.)—If $\Sigma_r = 1^r+2^r \dots +n^r$, prove that
$$7\Sigma_6+5\Sigma_4 = 12\Sigma_2\Sigma_3.$$

9042. (H. L. Orchard, B.Sc., M.A.)—Prove that $1^3+2^3+3^3+\dots+x^3$ is a factor of the expression $3x^8+12x^7+14x^6-7x^4+2x^2$.

9102. (H. L. Orchard, B.Sc., M.A.)—Show that the series
$$1^7+2^7+3^7+4^7+\dots+9^7 \text{ is divisible by } 27.$$

8647. (R. W. D. Christie, M.A.) — If $s = 1^3+2^3+3^3+\dots+n^3$, $S = 1^5+2^5+3^5+\dots+n^5$, $\Sigma = 1^7+2^7+3^7+\dots+n^7$; prove that $\Sigma+S = 2s^2$.

9142. (R. W. D. Christie, M.A. See Quest. 8700.)—If
$$\Sigma_r = 1^r+2^r+3^r \dots\dots n^r,$$
prove that
$$(9\Sigma_{11}+30\Sigma_9+9\Sigma_7)/\Sigma_3 = (11\Sigma_{10}+30\Sigma_8+7\Sigma_6)/\Sigma_2.$$

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*Solution.**

In 1834 Jacobi proved that S_{2n+1} contained S_1^2 as a factor, and that S_{2n} contained S_1 as a factor. Another proof was given by Prouhet in 1851; and again, an *à priori* proof by Cayley in 1857; and there are probably others. The simplest of those mentioned is Prouhet's; viz., writing $(1+h)^r$ in the form $1+r_1h+r_2h+\dots$,

it is easily shown that $(S_r = 1^r+2^r+3^r+\dots+n^r)$;

$$(r = \text{odd}),\quad r_1S_{r-1}+r_3S_{r-3}+\dots+r_{r-4}S_4+r_{r-2}S_2 = \tfrac{1}{2}[(n+1)^r+n^r-2n-1] \quad\dots\dots\dots (1);$$

$$(r = \text{even}),\quad r_1S_{r-1}+r_3S_{r-3}+\dots+r_{r-3}S_3+r_{r-1}S_1 = \tfrac{1}{2}[(n+1)^r+n^r-1] \quad\dots\dots\dots (2).$$

In (2), $(n+1)^r+n^r-1$ vanishes when $n=0$ and when $n+1=0$; therefore, by putting $r = 2, 4, 6, \dots$, we prove that S_1, and therefore S_3, and therefore S_5, and so on, are successively divisible by $\frac{1}{2}n(n+1)$. But Prouhet proved the full theorem. Let

$$K = (n+1)^r+n^r-1-2rS_1 = (n+1)^r+n^r-1-rn(n+1);$$

then both K and dK/dn vanish when $n=0$ and when $n+1=0$, therefore K contains $n^2(n+1)^2$ as a factor. Hence, by putting $r = 4, 6, 8 \dots$, we prove successively that $S_3, S_5, S_7 \dots$ contain S_1^2 (or S_3) as a factor. Similarly, the second part of the theorem is proved. (No. 9643.)

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* This solution is due to Mr. J. D. H. Dickson.

The following theorem is capable of proof—

$$(2n-1)\,S_{2n-2} = 3S_2\left\{a_0S_{2n-5}-a_2S_{2n-7}+a_4S_{2n-9}-\ldots\pm a_{2n-6}S_1\mp a_{2n-5}\right\} \quad \ldots\ldots\ldots(3),$$

$$2n\,S_{2n-1} = 4S_3\left\{b_0\,S_{2n-5}-b_2S_{2n-7}+b_4S_{2n-9}-\ldots\pm b_{2n-6}S_1\mp b_{2n-5}\right\} \quad \ldots\ldots\ldots(4),$$

where $\quad a_0 = 2n-4, \qquad\qquad b_0 = 2n-4,$

$\qquad\qquad a_2 = \frac{1}{4}(2n-4)(2n-6), \qquad b_2 = \frac{1}{3}(2n-4)(2n-6),$

the remaining a's and b's being somewhat complicated functions of Bernoulli's numbers. The first few cases are appended—

$$5S_4 = 3S_2\left\{2S_1-\tfrac{1}{3}\right\} \quad \ldots\ldots (5),$$

$$6S_5 = 4S_3\left\{2S_1-\tfrac{1}{2}\right\} \quad \ldots\ldots (6),$$

$$7S_6 = 3S_2\left\{4S_3-2S_1+\tfrac{1}{3}\right\} \quad \ldots\ldots (7),$$

$$8S_7 = 4S_3\left\{4S_3-\tfrac{8}{3}S_1+\tfrac{2}{3}\right\} \quad \ldots\ldots (8),$$

$$9S_8 = 3S_2\left\{6S_5-6S_3+\tfrac{18}{5}S_1-\tfrac{3}{5}\right\} \quad \ldots\ldots (9),$$

$$10S_9 = 4S_3\left\{6S_5-8S_3+6S_1-\tfrac{3}{2}\right\} \quad \ldots\ldots (10),$$

$$11S_{10} = 3S_2\left\{8S_7-12S_5+16S_3-10S_1+\tfrac{5}{3}\right\} \quad \ldots\ldots (11),$$

$$12S_{11} = 4S_3\left\{8S_7-16S_5+26S_3-20S_1+5\right\} \quad \ldots\ldots (12),$$

$$13S_{12} = 3S_2\left\{10S_9-20S_7+44S_5-\tfrac{438}{7}S_3+\tfrac{1382}{35}S_1-\tfrac{691}{105}\right\} \quad \ldots\ldots (13),$$

$$14S_{13} = 4S_3\left\{10S_9-\tfrac{80}{3}S_7+\tfrac{1052}{15}S_5-\tfrac{1786}{15}S_3+\tfrac{1382}{15}S_1-\tfrac{691}{30}\right\} \ldots (14),$$

&c.

No. 9226 follows from equation (6).

No. 8784 (the same as 9668) may be written in the form

$$(3S_5+2S_1S_3)/S_3 = 5S_4/S_2;$$

and by equations (5) and (6) each side equals $6S_1-1$.

No. 9683 comes from (5) and (7) by simple addition.

No. 9042 is "prove that $24S_7$ = multiple of S_3." And No. 9102 is nearly the same question, with 9 written for x.

No. 8647 is $S_5+S_7 = 2S_3^2$, and, like No. 9683, follows from equations (6) and (8).

No. 9142, by equations (8), (10), (12), and (7), (9), (11), shows that each side of the given relation is equal to $24\,(S_7+S_5)$.

The number of relations like the above may be indefinitely extended by the theorems (3) and (4).

9767. (R. W. D. Christie.)—Prove that n^m is the sum of n consecutive odd numbers.

Solution.

$n^{m-1}-n$ is always even $\equiv 2p$, suppose.

Then $\quad n^m = 2pn+n^2$

$\qquad\qquad = $ the sum of $2p+1,\ 2p+3,\ 2p+5,\ \ldots$ to n terms.

Thus also

$$\Sigma_n^2 = (m\,.\,m+1\,.\,2m+1)/6 = n+3\,(n-1)+5\,(n-2)\ldots 2m-1.$$

9876. (R. W. D. Christie.)—Prove that

$$2\tan^{-1}\frac{a}{b} \pm \tan^{-1}\frac{1}{b^2+2ab-a^2} = \tfrac{1}{4}\pi,$$

where a is the coefficient of x^n and b of x^{n+1} in the expansion of $\dfrac{1}{1+2x-x^2}$.

Solution.

If $2ab \pm 1 = b^2 - a^2$, then

$$2\tan^{-1}\frac{a}{b} \pm \tan^{-1}\frac{1}{b^2+2ab-a^2} = \tfrac{1}{4}\pi$$

becomes $$\tan^{-1}\frac{2ab}{2ab\pm 1} \pm \tan^{-1}\frac{1}{4ab\pm 1} = 1.$$

Now the coefficient of x^n in the expansion of $\dfrac{1}{1+2x-x^2}$ is the sum of $(n+1)$ terms of

$$\left\{ 2^n + (n-1)\,2^{n-2} + \frac{n-2\,.\,n-3}{2!}\,2^{n-4} \ldots\ldots n+1 \text{ or } 1 \right\}(-1)^n,$$

and these coefficients bear the assumed relation; the sign depending on b.

Examples.—If $n = 2$, then $a = 5$ and $b = 12$; thus we have

$$2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239} = \tfrac{1}{4}\pi,$$

which is Machin's formula.

If $n = 5$, then $a = 70$ and $b = 169$; thus we obtain

$$2\tan^{-1}\frac{70}{13^2} + \tan^{-1}\frac{1}{47321} = \tfrac{1}{4}\pi.$$

(E.) NEW QUESTIONS.

9877. Diophantus' Epitaph.

Hic Diophantus habet tumulum, qui tempora vitae
Illius mirâ denotat arte tibi.
Egit sextantem juvenis; lanugine malas
Vestire hinc coepit parte duodecimâ.

Septante uxori post haec sociatur, et anno
Formosus quinto nascitur inde puer.

Semissem aetatis postquam attigit ille paternae
Infelix subitâ morte peremptus obit.
Quatuor aestates genitor lugere superstes
Cogitur: hinc annos illius assequere.

9878. Every number contains an even number of factors, and therefore the numbers of odd and of even factors are either both odd or both

even, except when the original number is a square, and then the reverse is the case (*i.e.*, it contains an odd number of odd factors, and an even number of even factors, and consequently an odd number of factors).

9879. Prove that $a^r b^r c^r \ldots$ (where $r = 2, 3$, or 4) is of the same form as the squares, cubes, and biquadrates themselves, viz., $4N$ and $4N+1$, $9N$ and $9N \pm 1$, $16N$ and $16N+1$.

9880. 1. If $N = a^2 + b^2$, prove that it also equals

$$\left\{2mnb + (n^2 - m^2)\, a\right\}^2 / (m^2+n^2)^2 + \left\{2mna + (m^2 - n^2)\, b\right\}^2 / (m^2+n^2)^2,$$

where a, b, m, n, are any integers whatever.

9881. Show that the sum of n terms of the following n series

$$1^r + 2^r + 3^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ n^r,$$
$$1^r + 3^r + 5^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2n-1)^r,$$
$$1^r + 4^r + 7^r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3n-2)^r,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$1^r + (n+1)^r + (2n+1)^r \ldots\ldots\ (n^2-n+1)^r,$$

is $$= \Sigma_n^r \Sigma_{n-1}^r + n\Sigma_n^{r-1}\Sigma_{n-1}^{r-1} + \frac{n \,.\, n-1}{1 \,.\, 2}\Sigma_n^{r-2}\Sigma_{n-1}^{r-2} + \ldots + n^2,$$

where $$\Sigma_n^r = \text{sum of } 1^r + 2^r + 3^r \ldots\ldots \text{ to } n \text{ terms}.$$

9882. Let $s = 1^2 + 2^2 + 3^2 \ldots\ldots n^2$, $S = 1^3 + 2^3 + 3^3 \ldots\ldots n^3$,
$\Sigma = 1^5 + 2^5 + 3^5 \ldots\ldots n^5$, then $S + 2\Sigma = 3s^2$.

9883. Prove the following property of prime numbers. Distribute the primes from unity together with their multiples as in Quest. 9225, into groups of *four* (having however *five* in the *first* group) then

$$g_n = t + u = 6\,(g+2),$$

where g means the group, t the tens, and u the units in any prime.

Ex. gr.,
$$g_1 = (1+2+3+5+7) = 6 \times 3,$$
$$(11,\ 13,\ 17,\ 19) = 2 + 4 + 8 + 10 = 6 \times 4,$$
$$(23,\ 29,\ 31,\ 37) = 5 + 11 + 4 + 10 = 6 \times 5,$$
$$[41,\ 43,\ 47,\ (49)] = 5 + 7 + 11 + 13 = 6 \times 6, \text{ and so on.}$$

9884. Prove that the sum of the factors of any number is

$$S_f = (2^{n+1} - 1)\left(\frac{a^f - 1}{a-1}\right), \quad a = \text{any prime} > 2,$$

where f (a prime) the number of odd factors is $1/n^{\text{th}}$ of the number of even factors. Show also that, if n is any even number, N is a square (except when $f = 2$). [*Ex.*—Let N have 7 odd and 14 even factors. Then $S_7 = (2^3 - 1)\left(\frac{a^7-1}{a-1}\right)$, and $N =$ a square.]

9885. Let $\sigma = 1^2 + 2^2 + 3^2 \ldots\ldots n^2$, $s = 1^3 + 2^3 + 3^3 \ldots\ldots n^3$,
$S = 1^4 + 2^4 + 3^4 \ldots\ldots n^4$, $\Sigma = 1^6 + 2^6 + 3^6 \ldots\ldots n^6$.

Then $$7\Sigma + 5S = 4s \times 3\sigma.$$

9886. Every square number is divisible into two sequences from m any integer).

9887. Prove the following equation

$$a_\sigma^{p+q} = a^q\,(a_\sigma^p) + a_\sigma^{q-1},$$

where a means any prime number, and a_σ^p = sum of factors of a^p; hence show, if we could solve $2 = \dfrac{a^q\,(a_\sigma^p) + a_\sigma^{q-1}}{a^{p+q}}$ in integers, an odd perfect would be found.

9888. Divide the sum of two cubes into two other cubes.

9889. Take any number of my digits (1, 2, or 3 together), and I am equal to a sequence from unity. Cast out the nines from my dozen divisors and you'll find the factors of each of my digits. I am a famous number, but not a perfect number, and both myself and the sum of my digits are divisible by a perfect number.

9890. Find a number from the remainders after dividing it by a number of primes, say 3, 5, 7, 11, and 13.

9891. Draw a straight line cutting two concentric circles, so that the part intercepted by them is divided into three equal portions.

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